Compactification of a Non-compact topological space

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***Abstract*: In this paper, we have proved in details the existence of a homeomorphism between $\mathbb{R}^n$ compactified and a topological variety in n+1 dimensions, knowing that the $\mathbb{R}^n$ topological space is not compact but locally compact.**

***Index Terms*: Compactification, Locally Compact Space, Homeomorphism, Alexandrov Compactification, Stereographic Projection.**

I. INTRODUCTION

The compactification of a non-compact topological space X that is locally compact consists to find a topological space $\hat{X}$ by including to X an infinite point. Such that $\hat{X}$ becomes a compact space containing X as dense subspace and $\hat{X}$ makes a homeomorphism with on other known topological space [7].

The meaning of the concept of compactness stems from the fact that it makes it possible to reduce problems of infinite complexity to the study of a finite number of cases [2].

The compactification notions are introduced in [6] and [10], but in this article we makes in more details the compactification of $\mathbb{R}^n$, $n \geq 1$ with his topology. Starting by $\mathbb{R}$ and $\mathbb{R}^2$, we generalized to $\mathbb{R}^n$. We found and proved the existence of a homeomorphism between the compacfied $\widehat{\mathbb{R}}^n$ and the $\mathbb{R}^{n+1}$ sphere, in the following lines.

II. PRELIMINARIES AND NOTATIONS

We give differents definitions and theorems that will be important in the below.

Definition 2.1 (homeomorphism, [9])

Let (X, τ) and (Y, μ) be two spaces topological. A function $f: X \to Y$ is an homeomorphism if f is bijective and bicontinous (continue and opened).

Theorem 2.1. [3]

Let (X, τ) and (Y, μ) be two spaces topological. A continue function $f: X \to Y$ is a homeomorphism if and only it exists a continue function $g: X \to X$ such that $gof = id_X$ and $fog = id_Y$.

Definition 2.2. (Locally compact space, [3], [8])

A topological space X is locally compact wheter it is separated and each point of X has a compact neighborhood.

Definition 2.3. (compactification, [6])

Let (X, τ) be a topological space. We call compactified of X, each couple $(\hat{X}, h)$ such that:

$i)$ $\hat{X}$ is a compact space

$ii)$ $h: X \to \hat{X}$ is a continue injection inducing an homomorphism $h: X \to h(x)$.

$iii)$ $\overline{h(x)} = \hat{X}$ in which $\overline{h(x)}$ is the adherence of $h(x)$. In other words, the compactified of X, denoted $\hat{X}$, is a compact space containing X as a dense subspace.

Theorem 2.2. (Alexandrov compactification, [5], [7])

Any locally compact spaces $(X, \hat{o})$ admits a compactified $\hat{X}$ as $\hat{X} \backslash X$ is a single set.

Proof.

Let $\hat{X} = X \cup \{\infty\}$ and

$\hat{\hat{o}} = \hat{o} \cup \{(X/K) \cup \{\infty\}; K\ compact\ \subseteq k\}$. We must prove that:

$i)$ $\hat{\hat{o}}$ is a topology on $\hat{X}_1$,

$ii)$ $(\hat{X}, \hat{\tau})$ is compact,

$iii)$ and $h: X \to \hat{X}$continue.

Let us prove above items:

$i)$ $\emptyset \epsilon \tau \subseteq \hat{\tau} \Rightarrow \emptyset \epsilon \tau$;

$\hat{X} = (X \backslash \emptyset) \cup \{\infty\} \in \hat{\tau}, \emptyset$ a compact.

$Let\ \vartheta_1\ and\ \vartheta_2 \in \tau;$

- $if\ \vartheta_1\ and\ \vartheta_2 \in \tau \Rightarrow \vartheta_1 \cap \vartheta_2 \in \tau \subseteq \hat{\tau}.$
- $if\ \vartheta_1, \vartheta_2 \epsilon\ \hat{\tau} \backslash \tau \Rightarrow \exists\ K_1\ and\ K_2$.

two compacts *include in* X such that

$$\vartheta_i = (X\setminus K_i) \cup \infty \quad with \quad i = 1,2.$$
$$\Rightarrow \vartheta_1 \cap \vartheta_2 = (X\setminus(K_1 \cup K_2) \cup \{\infty\} \epsilon \hat{\tau}$$

$if\ \vartheta_1 \in \tau\ ,\ \vartheta_2 \in \hat{\tau}\setminus\tau \Rightarrow \vartheta_2 = (X\setminus K) \cup \{\infty\},$
Kcompact if $X \Rightarrow \vartheta_1 \cap \vartheta_2 = \vartheta_2 \cap (X/K) \cup \{\infty\} = \vartheta_1 \cap (X\setminus K)$.
Because K is closed. According that K is a compact in the separated space X.
Let $\{\vartheta_i\}_{i\epsilon\tau}, \vartheta_i\epsilon\hat{\tau}$.

- $if\ \forall\ i\ \in J, \vartheta_i \in \tau\ \Rightarrow \cup_{i\in J} \vartheta_1 \in \tau \subseteq \hat{\tau}$
- $if\ \forall\ i\ \in J, \vartheta_i\epsilon\hat{\tau}\setminus\tau \Rightarrow \vartheta_i = (X\setminus K_i) \cup \{\infty\}\ , K_i compact$

$$\bigcup_{i\in J} \vartheta_i = X\setminus (\cap_{i\epsilon\tau} K_i) \cup \{\infty\} \in \hat{\tau}$$

- $if\ J = J_1 \cap J_2$ as $\forall\ i \in J_1,\ \vartheta_i \in \tau; \forall i\ \in J_2, \vartheta_1 = (X\setminus K_i) \cup \{\infty\};$

$\cup_{i\epsilon\tau} \vartheta_i = (\cup_{i\epsilon\tau} \vartheta_i) \cup (\cup_{i\epsilon\tau} (X\setminus K_i) \cup \{\infty\})= (\cup_{i\in?} \vartheta_i) \cup (X\setminus(\cap_{i\epsilon\tau} K_i) \cup \{\infty\}) \in \hat{\tau}$
which shows that $\hat{\tau}$ is a topology on $\hat{X}$.

$ii)\big(\hat{X}, \hat{\tau}\big)$ is compact;

- $\hat{X}$ is separated if $x, y \in \hat{X}$

$$x \neq y \begin{cases} if\ x, y \in X\ because\ X\ is\ separeted \\ if\ x, y \in X, y = \infty, X\ locally\ compact \Rightarrow x\ admits\ a\ compact\ neighborhood\ A. \end{cases}$$

So, $B = (X\setminus A) \cup \{\infty\}$ is neighborhood of y and $A \cap B = \emptyset$.

- $\hat{X}$ verifies Lebesgue – Borel Axiom.
  Let $\{\vartheta_i\}_{i\epsilon\tau}$ an opened recovery of $\hat{X}$, $\vartheta_i \in \hat{\tau}$,
  $$\hat{X} = \cup_{i\epsilon\tau} \vartheta_i \Rightarrow \exists i_0 \in \tau{:}\, \infty \in \vartheta_{io} = (X\setminus K) \cup \{\infty\},$$
  Kcompact of X.
  $K \subseteq \hat{X} = \cup_{i\epsilon\tau} \vartheta_i, K\ compact \Rightarrow \exists i_1, \dots .\, i_q$ as $K \subseteq \vartheta_{i1} \cup \dots \cup \vartheta_{iq}$, so $\vartheta_{i1}, \vartheta_{i2}, \dots\ \vartheta_{iq}$ a finite subrecovery of X.

$iii)$ $h{:}\, X \to \hat{X}$ a canonical injection; *is* h continue?
Let $u \in \hat{\tau} \Rightarrow h^{-1}(u) \in \tau$.

- If $u \in \tau \Rightarrow h^{-1}(u) = u \in \tau$
- If $u \in \hat{\tau}\setminus\tau \Rightarrow h^{-1}(u) = h^{-1}\big((X\setminus K) \cup \{\infty\}\big) = X\setminus K \in \tau\ \Rightarrow h$ continue.

(iv) $\hat{X} = \bar{X}$. it is enough to show that $\infty \in \bar{X}$;
Indeed, let V be a neighborhood of $\infty \Rightarrow V \supseteq (X\setminus K) \cup \{\infty\}$ and we have:

$$V \cap X = X\setminus K \neq \emptyset.$$

Hence $\hat{X}$ is a compactified of X and we also have:

$$\hat{X}\setminus X = \{\infty\} \text{ (an infinite point).}$$

III. COMPACTIFICATION OF $\mathbb{R}^n$ SPACE

Case 1: n= 1

Let the topological space $\mathbb{R}$. $\mathbb{R}$ is not compact but $\mathbb{R}$ is locally compact because $\forall x\epsilon\mathbb{R}, [x-1, x+1]$ is a compact neighborhood of x.

The $\mathbb{R}$ compactified, denoted $\hat{\mathbb{R}}$ is the achieved line $\bar{\mathbb{R}}= \mathbb{R} \cup \{\infty\}$.

We will prove that; this compactified $\bar{\mathbb{R}}$ is homeomorphic to a circle centered in $C = (0,1)$ point of radius 1, that we denote C^1.

$$C^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + (y-1)^2 = 0\} \qquad (1)$$

thereby,

Let us consider an application $h{:}\, C^1 \to \mathbb{R} \cup \{\infty\}$; let us define the h application:

let $P(x, y)$ be a certain point and $N(0,2)$ a point of C^1 circle.

The parametric equation of NP line is $(1-\lambda)N + \lambda P$.

Then,

$$(1-\lambda)(0,2) + \lambda(x, y) = (\lambda x, 2(1-\lambda) + \lambda y).$$

The image point of $P(x, y)$ by h application is the intersection of NP line and $x - axis$. So we have:

$$2(1-\lambda) + \lambda y = 2 - 2\lambda + \lambda y = 0 \Rightarrow 2 - \lambda(2-y) = 0$$
$$\Rightarrow \lambda = \frac{2}{2-Y}.$$

Let us define h:

$$C^1 \to \mathbb{R} \cup \{\infty\},$$

$$\text{by } h(P) = h(x,y) = \begin{cases} \frac{2x}{2-Y} \ \forall\, P(x,y) \neq N \\ \infty \ for\ (x,y)=(0,2)=N \end{cases};$$

h application is well defined because it takes all of it values in $\mathbb{R} \cup \{\infty\}$.

It sufficient to show that h is an homeomorphism. It is clear to see that h is continue. Let us find a h^{-1}:$\mathbb{R} \cup \{\infty\} \to C^1$ application that must be continue and more

$$h^{-1}oh = id_{C^1} \text{ and } hoh^{-1} = id_{\mathbb{R}\cup\{\infty\}}.$$

Let $x \in \mathbb{R} \cup \{\infty\}$. The h^{-1} image of this x point will be the meet of the C^1 circle with the line reaching by $N(0,2)$ and $(x,0)$, x point considered in the plane.

The NP line parametric equation is:

$$(1-x)(0,2) + \lambda(x,0) = \big(\lambda x, 2(1-\lambda)\big). \quad (2)$$

This $\big(\lambda x, 2(1-\lambda)\big)$ point would be on the C^1 circle that verifies:

$$(\lambda x)^2 + (2(1-\lambda)-1)^2 - 1) = 0$$

$$\Longrightarrow \lambda^2 x^2 + 4 - 8\lambda + 4\lambda^2 - 4 + 4\lambda = 0$$

$$\Longrightarrow \lambda^2 x^2 - 4\lambda + 4\lambda^2 = O \Longrightarrow \lambda^2(x^2+4) - 4\lambda = 0 \Rightarrow \lambda = \frac{4}{x^2+4}$$

We have, $h^{-1}(x) = \left(\frac{4x}{x^2+4}, 2\left(1 - \frac{4}{x^2+4}\right)\right) = \left(\frac{4x}{x^2+4}, \frac{2x^2}{x^2+4}\right)$.

The problem will be to prove that h^{-1} is well defined,

$$i.e.\ h^{-1}(x) = \left(\frac{4x}{x^2+4}, \frac{2x^2}{x^2+4}\right) \in C^1,$$

in other words, coordinates of this point must verify the C^1 equation:

$$\left(\frac{4x}{x^2+4}\right)^2 + \left(\frac{2x^2}{x^2+4} - 1\right)^2 = 1.$$

Indeed, $\left(\frac{4x}{x^2+4}\right)^2 + \left(\frac{2x^2}{x^2+4} - 1\right)^2 = 1?$

Finally, h^{-1}: $\mathbb{R} \cup \{\infty\} \to C^1$ is well defined by $h^{-1}(x) = \begin{cases} \left(\frac{4x}{x^2+4}, \frac{2x^2}{x^2+4}\right) \ \forall x \in \mathbb{R} \\ N(0,2) \quad for\ x = \infty \end{cases}$.

Thereafter, this h^{-1} application is continue.

Ultimately, let us prove that $h^{-1}oh = id_{C^1}$ and $hoh^{-1} = id_{\mathbb{R}\cup\{\infty\}}$

$$\forall (x,y) \in C^1, (h^{-1}oh)(x,y) = \left(\frac{4\left(\frac{2x}{2-y}\right)}{\left(\frac{2x}{2-y}\right)^2 + 4}, \frac{2\left(\frac{2x}{2-y}\right)^2}{\left(\frac{2x}{2-y}\right)^2 + 4}\right).$$

The first component

$$\frac{4\left(\frac{2x}{2-y}\right)}{\left(\frac{2x}{2-y}\right)^2 + 4} = x$$

The second component

$$\frac{2\left(\frac{2x}{2-y}\right)^2}{\left(\frac{2x}{2-y}\right)^2 + 4} = y.$$

Involving $hoh^{-1} = id_{\mathbb{R}\cup\{\infty\}}$. Thereby h is an homeomorphism.

Case 2: n=2

The compactified of $\mathbb{R}^2$ is $\mathbb{R}^2 \cup \{\infty\}$ and let us prove that this compactified is homeomorphic to the $\mathbb{R}^3$ sphere centered to the (0,0,1) point and the radius is 1, denoted by C^2.

$$C^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + (z-1)^2 - 1 = 0\} \quad (3)$$

Let $P(x,y,z)$ be a point of C^2. The NP line equation is:

$$(1-\lambda)(0,0,2) + \lambda(x,y,z) = (\lambda x, \lambda y, 2(1-\lambda) + \lambda z)$$

where $N(0,0,2)$ is the north pole .

The intersection of NP line and XoY plane:

$2(1-x) + \lambda z = 0$. That intersection is given by $\left(\frac{2x}{2-y}, \frac{2y}{2-z}, 0\right)$ and we have the application:

h: $C^2 \to \mathbb{R}^2 \cup \{\infty\}$ defined by:

$$h(x,y,z)=\begin{cases}\left(\frac{2x}{2-y},\frac{2y}{2-3}\right)\ for\ all\ (x,y,z)\neq N\\ \infty\ for\ (x,y,z)=(0,0,2)=N\end{cases}\qquad(4)$$

h is continue because is defined in one critical point who is the north pole $N\ (0,0,2)$. This application is called stereographic projection.

Let us find the reciprocal function $h^{-1}:\mathbb{R}^2\cup\{\infty\}\to C^2$ giving the image of $(x,y),\mathbb{R}^2$ point by intersection of the line that joining this point with the north pole N and the sphere:

$$(1-\lambda)(0,0,2)+\lambda(x,y,0)=(\lambda x,\lambda y,2(1-\lambda)).$$

That point belongs to C^2 if and only his coordinates verify the C^2 equation. So we have,

$$(\lambda x)^2+(\lambda y)^2+(2(1-\lambda)-1)^2-1=0,$$

involving $\lambda=\frac{4}{x^2+<^2+4}$.

$$h^{-1}(x,y)=\left(x\frac{4}{x^2+y^2+4},y\frac{4}{x^2+y^2+4},2\frac{4}{x^2+y^2+4}\right)$$

Or $h^{-1}(x,y)=\left(\frac{4x}{x^2+y^2+4},\frac{4y}{x^2+y^2+4},\frac{2(x^2+y^2)}{x^2+y^2+4}\right)$,

so that

$$h^{-1}(x,y)=\begin{cases}\left(\frac{4x}{x^2+y^2+4},\frac{4y}{x^2+y^2+4},\frac{2(x^2+y^2)}{x^2+y^2+4}\right)\forall(x,y)\in\mathbb{R}^2\\ N(0,0,2)for\ the\ \infty\ point\end{cases}\qquad(5)$$

we must now prove that this h^{-1} is well defined,

$$\text{i.e. } h^{-1}(x,y)=\left(\frac{4x}{x^2+y^2+4},\frac{4y}{x^2+y^2+4},\frac{2(x^2+y^2)}{x^2+y^2+4}\right)\in C^2$$

in other words, coordinates of this point must verify C^2 equation.

First we must show this:

$$\left(\frac{4x}{x^2+y^2+4}\right)^2+\left(\frac{4y}{x^2+y^2+4}\right)^2+\left(\frac{2(x^2+y^2)}{x^2+y^2+4}-1\right)^2=1.$$

Indeed,

$$\left(\frac{4x}{x^2+y^2+4}\right)^2+\left(\frac{4y}{x^2+y^2+4}\right)^2+\left(\frac{2(x^2+y^2)}{x^2+y^2+4}-1\right)^2=\frac{x^4+y^4+8x^2+8y^2+2x^2y^2+16}{x^4+y^4+8x^2+8y^2+2x^2y^2+16}=1.$$

Thereby,

$$\forall x\in\mathbb{R}^2,(hoh^{-1})(x,y)=h\big(h^{-1}(x,y)\big)=h\left(\frac{4x}{x^2+y^2+4},\frac{4y}{x^2+y^2+4},\frac{2(x^2+y^2)}{x^2+y^2+4}\right)$$
$$=(x,y)=id_{\mathbb{R}^2\cup\{\infty\}}(x,y).$$

In the other hand, $\forall(x,y,z)\in C^2$,

$$(h^{-1}oh)(x,y,z)=h^{-1}(h(x,y,z)=h^{-1}\left(\frac{2x}{2-y},\frac{2y}{2-z}\right)$$
$$=\left(\frac{8x(2-x)}{4(x^2+y^2+z^2-4z+4)},\frac{8y(2-x)}{4(x^2+y^2+z^2-4z+4)},\frac{8x^2+8y^2}{4(x^2+y^2+z^2-4z+4)}\right)$$

and

$$(h^{-1}oh)(x,y,z)=\left(\frac{8x(2-x)}{4(-2z+4)},\frac{8y(2-x)}{4(-2z+4)},\frac{8x^2+8y^2}{4(-2z+4)}\right)(x,y,z)=id_{C^2}(x,y).\qquad(6)$$

Thereupon, h is an homeomorphism.

IV. CONCLUSON

In this paper we gave this generalization: The $\mathbb{R}^n$ compactified is $\mathbb{R}^n\cup\{\infty\}$, that is homeomorphic to the C^n sphere centered on $(0,0,\dots,0,1)$ point and his equation is

$$C^n=\{x_1,x_2,\dots,x_n,x_{n+1})\in\mathbb{R}^{n+1}|x_1^2+x_2^2+\cdots+x_n^2+(x_{n+1}-1)^2-1=0\}.$$

In the same way we found the stereographic projection:

$h:C^n\to\mathbb{R}^n\cup\{\infty\}$ defined by:

$$h(x_1,x_2,\dots,x_n,x_{n+1})=\left(\frac{x_1}{2-x_{n+1}},\frac{x_2}{2-x_{n+1}},\dots,\frac{x_n}{2-x_{n+1}}\right)$$

and we gave his reciprocal function:

$$h^{-1}(x_1,x_2,\dots,x_n)$$
$$=\begin{cases}\left(\frac{4x_1}{x_1^2+x_2^2+\cdots+x_n^2+4},\frac{4x_2}{x_1^2+x_2^2+\cdots+x_n^2+4},\dots\frac{2(x_1^2+x_2^2+\cdots+x_n^2)}{x_1^2+x_2^2+\cdots+x_n^2+4}\right)\ \forall\ (x_1,x_2,\dots,x_n)\\ N(0,0,\dots,2)\ on\ \infty\ point\end{cases}$$

h and h^{-1} are continue and well definied as $h^{-1}oh=id_{C^n}$ and $hoh^{-1}=id_{\mathbb{R}^n\cup\{\infty\}}$. Therefore h is an homeomorphism.

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