The Ito Formula and the Distributions Theory

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***Abstract*: In this paper, we have achieved the generalization of Ito formula, by considering his C^2 function f like a distribution (extending differentiability conditions on this f function). For this, we used the Tanaka's theorem in $\mathbb{R}$; and the Brosamler's theorem in $\mathbb{R}^n$. And then, the possibility to use a C^n function in which the last part that is the second derivative of the f function (in distributions term) will be a functional.**

***Keywords*: Ito Formula, Brownian motion, Distribution, Tanaka's theorem, Ito-Tanaka Formula, Brosamler's Theorem.**

I. INTRODUCTION

Let f be a function of $C^2(\mathbb{R})$ and let X_t be a brownian motion in $\mathbb{R}^n$, the Ito formula is :

$$f(X_t) = f(X_O) + \int_o^t f'(X_s)dX_S + \frac{1}{2}\int_0^t f''(X_s)d_s . \qquad (1)$$

The equation (1) can be extended in two dimensions like this:

Let f be function in $([0,T] \times \mathbb{R})$, f is C^1 class compare to $t \in ([0,T])$, f is C^2 compare to $x (x \in \mathbb{R})$, we have

$$f(t, X_t) = f(0, X_0) + \int_0^t f_t'(s, X_s)ds + \int_0^t f_x'(s, X_s)dX_s + \frac{1}{2}\int_0^t f_{xx}''(s, X_s)d\langle X\rangle_s$$

his infinitesimal notation is:

$$df(t, X_t) = f_t'(s, X_s)ds + f_x'(s, X_s)dX_s + \frac{1}{2} f_{xx}''(s, X_s)d\langle X\rangle_s$$

where $\langle X\rangle_s$ is the quadratic variation of X (It can be interpreted like the length of X trajectory). For other forms of Ito formula see [4] and [13].

Let us try to modify differentiability conditions of f function in (1) (by assuming that f is a distribution) and let us do the interpretation of the last part.

II. DISTRIBUTIONS THEORY

The distributions set into an opened set of $\mathbb{R}^n$ is particular because it renders functions indefinitely differentiable in « distributions term » on the same opened. That's why we will consider in the following, the f function in (1) like a distribution.

Definition 2.1 (Support of a function, [10])

Let u be a $\mathbb{R}^n$ in $\mathbb{C}$ function, we call support of u, and we denote $supp(u)$ the adherence in $\mathbb{R}^n$ of the set $A = \{x \in \mathbb{R}^n | \ u(x) \neq 0\}$; then, $supp(u) = \bar{A}$.

Proposition 2.1

It exists a numeric function φ defined in $\mathbb{R}^n$ verifying :

1. $\varphi \neq 0 \ et \ \varphi \geq 0$;
2. $\varphi \in C^\infty(\mathbb{R}^n)$;
3. $supp(\varphi) \subset \{x \in \mathbb{R}^n | \ \|x\| \leq 1\}$
4. $\int_{\mathbb{R}^n} \varphi(x)dx = 1$

Definition 2.2 (regularizing sequence**,** [14]**)**

Let φ be a function that verifies hypothesis in the proposition 2.1; we call regularizing sequence associated to φ, the sequence of functions $(\varphi_k)_{k \in \mathbb{N}^*}$ defined by

$$\varphi_k : \mathbb{R}^n \to \mathbb{R}, x \to \varphi_k(x) = k^n \varphi(kx).$$

We have $\varphi_k \geq 0, \varphi_k \in C^\infty(\mathbb{R}^n)$

$$supp(\varphi_k) \subset B\left(0, \frac{1}{k}\right)$$

for each $k \in \mathbb{N}^*, \int_{\mathbb{R}^n} \varphi_k(x)dx = 1$

Definition 2.3 (order of a vector, [10])

Let $\boldsymbol{\Omega}$ be an opened set of $\mathbb{R}^n$, K designates a compact subset of $\mathbb{R}^n$, non-empty interior, include in $\boldsymbol{\Omega}$: $\emptyset \neq K^\circ \subset K \subset \boldsymbol{\Omega}$.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an element of $\mathbb{N}^n$. We call order of α and we denote $|\alpha|$ the integer:

$|\alpha| = \sum_{u=1}^n \alpha_i$.

Let u be a $\mathbb{R}^n$ function in $\mathbb{C}$, α an element of $\mathbb{N}^n$. We denote $D^\alpha u$ the α order derivative of u_α :

$$D^\alpha u = \frac{\partial^\alpha u}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n}$$

we denote $\mathfrak{D}_k(\Omega) = \{\boldsymbol{u} \in C^\infty(\Omega) |\ supp(u) \subset K\}$.

Definition 2.4 [14]

A sequence $(u)_{p\in\mathbb{N}}$ of $\mathfrak{D}_k(\Omega)$ converges towards u of $\mathfrak{D}_k(\Omega)$, and we denote $u = \lim_{p\to\infty}(u_p)$,

if $\forall \varepsilon, \forall k \epsilon \mathbb{N}, \exists_{p_o}, \forall p \geq p_0$: $\sup_{|\alpha|\leq k} \sup_{x\epsilon\Omega} \left|D^\alpha u(x) - D^\alpha u_p(x)\right| \leq \varepsilon$.

A subset A of $\mathfrak{D}_k(\Omega)$ is a bounded set if

$$\forall k \epsilon \mathbb{N}, \exists M_k > 0, \forall u \ \epsilon\ A, \sup_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha u(x)| < M_k$$

Definition 2.5 (Test functions space, [10]**)**

Let Ω be a non-empty opened set of $\mathbb{R}^n$.

We call test functions space and we denote $\mathfrak{D}_k(\Omega)$ the set in the below:

$$D(\Omega) = \{u\epsilon\ C^\infty(\Omega) | \exists K\ compact, K \subset \Omega, u \epsilon D_k(\Omega)\}.$$

Definition 2.6 (Distribution, [10] ,[14])

Let Ω an opened continuous linear form in $D(\Omega)$. And we denote $D'(\Omega)$ the distributions set.

Notation. For any (T, u) of $D'(\Omega) \times D(\Omega), T(u)$ belongs to $\mathbb{C}$, and we denote $T(u) = \langle T, u \rangle$.

Definition 2.7 (Derivatives of α order, [10])

Let $\boldsymbol{\Omega}$ be an opened set of $\mathbb{R}^n$, T an element of $D'(\Omega)$: for α of $\mathbb{N}^n$, we call derivative of order α of T and we denote $D^\alpha T$ the application :

$$D^\alpha : D(\Omega) \to \mathbb{C}, u \mapsto D^\alpha T(u) = (-1)^{|\alpha|} \langle T, D^\alpha u \rangle.$$

This concept of derivation will render indefinitely derivable any distributions; which will coincide by isomorphism with the derivative of C^1 and C^∞ class functions.

It allows to extend the derivation of $C^1(\Omega)$ class elements.

Whenever we will talk about distributions in the following, it will be within the meaning of the definition 2.6.

III. ITO FORMULA

Let us first define some concepts of stochastic processes that will help us to establish the Ito formula.

Definition 3.1 (Filtration, [1])

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probabilized space. A filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is the growing family of subtribes of $\mathcal{F}$ $(\mathcal{F}_0, \ldots, \mathcal{F}_T)$ such that for any $t \leq s, \mathcal{F}_t \leq \mathcal{F}_s$.

Definition 3.2 (adapted process, [5], [11])

Let $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ a filtration. A process $(X_t)_{t\in\mathbb{R}^+}$ is adapted if for any t, X_t is $\mathcal{F}_t$ – mesurable.

Definition 3.3 (Martingale, [6],[8])

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probabilised space and $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ filtration of this space an adapted collection $(M_t)_{t\in\mathbb{R}^+}$ of integrables random variables (verifying $\mathbb{E}(|\ M_t|\ ) < +\infty$, for any t) is a martingale if,

$$\forall s \leq t, \mathbb{E}(M_t|\mathcal{F}_t) = M_s.$$

Definition 3.4 (Semi-martingale, [7])
A stochastic process $(X_t)_{t\in\mathbb{R}^+}$ is a semi-martingale if X_t can be written in the form $X_0 + M_t + A_t$, *where* $M_0 = A_0 = 0$, M_t is a martingale and A_t is an adapted process.

Definition 3.5 (Brownian motion, [12])
A stochastic process $(X_t)_{t\in\mathbb{R}^+}$ is a Brownian motion if $(X_t)_{t\in\mathbb{R}^+}$ has an independent and stationary increments. It means that

- If $0 \leq s \leq t$, $X_t - X_s$ is independent of the tribe $\mathcal{F}_s = \sigma(X_u, u \leq s)$: Independence of increments,
- If $0 \leq s \leq t$, $X_t - X_s$ is identical to $X_{t-s} - X_s$ increments are stationary,
- $\forall\ \omega \in \Omega$ the relation s $\rightarrow X_s(\omega)$ is a function : continuity of trajectories.

Theorem 3.1 (Ito formula, [13], [1])
All function $f \in C^2(\mathbb{R})$ with second derivatives verifies:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)\,ds$$

$\forall t \leq T$, *where* X_t is a stochastic process.

Proof. See [13]

IV. GENERALIZATION OF THE ITO FORMULA

Apart integrability conditions, the formula (1) cannot be applied while the f function doesn't belong to C^2 class; let us suggest his generalization in the following manner:

Let us keep his stochastic process (Brownian motion) but let us relax differentiability hypothesis upon f function.

This kind of extension is given by Tanaka's formula. In $\mathbb{R}$, the Tanaka's formula is the Ito formula in which $f(x) = |x|$, that is not in C^2 class. Let us denote that f is C^2 class function in the complementary of open set of nul measure $\{0\}$, and his second derivative in distributions term is a measure. That means we would establish an Ito formula in which the last part must be interpreted.
For this, let us announce the following theorem:

Theorem 4.1 (Tanaka's Formula, [2])

Let X be a continuous semi martingale. It exists $(L_t^\alpha)_{t\geq 0}$, $\alpha \in \mathbb{R}$, a crescent continuous process called local time in α of the semi martingale X, such that :

$$(X_t - \alpha)^+ = (X_0 - \alpha)^+ + \int_0^1 1_{\{X_s > \alpha\}}\, dX_s + \frac{1}{2}L_t^\alpha,$$

$$(X_t - \alpha)^- = (X_0 - \alpha)^- - \int_0^1 1_{\{X_s \leq \alpha\}}\, dX_s + \frac{1}{2}L_t^\alpha$$

and

$$|X_t - \alpha| = |X_0 - \alpha| + \int_0^1 sgn\,(X_s - \alpha)dX_s + L_t^\alpha\ ; \qquad (2)$$

where $sgn(x) = -1\ or\ 1\ according\ that\ x \leq 0\ or\ x > 0.$

And more, the measure (of stieljes) dL_t^{α} associed to L_t^{α} is carried by $\{t \in \mathbb{R}: X_t = \alpha\}$.

Proof.

Let us consider φ a convex and continuous function. Though φ is not in C^2 class, let us try to write a Ito formula for (X_t) .

Let j be a positive C^{∞} class function with compact support includes in $]-\infty, 0]$ such that $\int_{-\infty}^{0} j(y)dy = 1$, Let us assume that

$$\varphi_n(x) = n\int_{-\infty}^{0} \varphi(x+y)j(ny)dy\,.$$

As φ convex and locally bounded, φ_n is well defined. And more φ_n is in C^{∞} class and just converges on φ and φ_n' grows towards φ_-', left derivative of φ. By applying Ito formula to the function we get

$$\varphi_n(X_t) = \varphi_n(X_0) + \int_0^t \varphi_n'(X_s)\,dX_s + \frac{1}{2}A_t^{\varphi_n} \qquad (3)$$

where $A_t^{\varphi_n} = \int_0^t \varphi_n''(X_s)d\langle X,X\rangle_s$,

we have $\lim\limits_{n\to+\infty} \varphi_n(X_t)$=$\varphi(X_t)$ et $\lim\limits_{n\to+\infty} \varphi_n(X_0)$=$\varphi(X_0)$.

We can assume that X et $\varphi_n'(X_s)$ are bounded (uniformly in n because $\varphi_1' \le \varphi_n' \le \varphi_-'$).

By the dominated convergence theorem of the stochastic integrals, we have:

$$\int_0^t \varphi_n'(X_s)\,dX_s \overset{\mathbb{P}}{\to} \int_0^t \varphi_-'(X_s)\,dX_s$$

uniformly on compacts. Therefore, A^{φ_n} converges towards a crescent process A^{φ} because it is a limit of crescent processes. By going to the limit (3), it happens this

$$\varphi(X_t) = \varphi(X_0) + \int_0^t \varphi_-'(X_s)dX_s + \frac{1}{2}A_t^{\varphi} \quad , \qquad (4)$$

so, the process A^{φ} can be chosen continuous (because it is a difference of continuous processes). Let us apply (4) to $\varphi(x) = (x-\alpha)^+$ a convex function with left derivative $\varphi_-' = 1_{]\alpha,+\infty[}$. It exists a crescent process A^+ such that

$$(X_t-\alpha)^+ = (X_0-\alpha)^+ + \int_0^t 1_{\{X_s>\alpha\}}dX_s + \frac{1}{2}A_t^+ \,. \qquad (5)$$

In the same manner with $\varphi(x) = (x-a)^-$ convex function of left derivative $\varphi_-' = -1_{]-\infty,\alpha]}$, it exists a crescent process A^- such that

$$(X_t-\alpha)^- = (X_0-\alpha)^- - \int_0^t 1_{\{X_s\le\alpha\}}dX_s + \frac{1}{2}A_t^- \,. \qquad (6)$$

As $x = x^+ - x^-$, we get from de difference between (5) and (6) :

$$X_t = X_0 + \int_0^t dX_s + \frac{1}{2}(A_t^+ - A_t^-) \,. \qquad (7)$$

It happens that $A^+ = A^-$ and we then pose $L_t^{\alpha} = A_t^+$.

By adding, as $|x| = x^+ + x^-$, we have

$$|X_t-\alpha| = |X_0-\alpha| + \int_0^t sgn(X_s-\alpha)\,dx_s + L_t^{\alpha}\,. \qquad (8)$$

For the last part, by applying the Ito formula to the semi-martingale $|X_t-\alpha|$ with $(x) = x^2$; we also get by using (7) :

$$|X_t-\alpha|^2 = |X_0-\alpha|^2 \int_0^t |X_s-\alpha| d(|X_s-\alpha|)_s$$

$$= (X_0-\alpha)^2 + 2\int_0^t |X_s-\alpha|\; sgn(X_s-\alpha)dX_s + 2\int_0^t |X_s-\alpha|\,dL_s^{\alpha} + \langle X,X\rangle_s\,;$$

by comparing with the Ito formula for x with $f(x) = (x-\alpha)^2$, we have :

$$(X_t-\alpha)^2 = (X_0-\alpha)^2 + 2\int_0^t (X_s-\alpha)dX_s + \langle X,X\rangle_s\,.$$

We get $\int_0^t |X_s-\alpha|\,dL_s^{\alpha} = 0$, which provides the desired result.

Theorem 4.2 (Ito-Tanaka Formula**,** [2]**)**

When $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a convex function, it is quite possible to specify (4).

Let us show that :

$$A_t^{\varphi} = 2\int_{-\infty}^{+\infty} L_t^{\alpha}\varphi''(d\alpha)\ .$$

Where $\varphi''(d\alpha)$ is the measure associated to φ'' (in distributions term). Then, we have the Ito-Tanaka formula for the convex function:

$$\varphi(X_t) = \varphi(X_0) + \int_0^t \varphi'_-(X_s)dX_s + \int_{-\infty}^{+\infty} L_t^{\alpha}\varphi''(d\alpha)\ . \qquad (9)$$

The same problem can be reformulated in $\mathbb{R}^n$ like this:

Given an opened set O, a C^2 class function f in O whose laplacian (second derivative in distributions term) is a measure, can we establish an Ito-Tanaka formula for f ?

R.K. Getoor and M.J. Sharpe in [15], tried to resolve this problem but with lightly larger conditions; N.V. Krylov has treated the same problem with f function belonging to sobolev space [9]; and G. Brosamler in [3], has also dealt with this problem in 1970. We are using Brosamler's work to finally conclude.

Theorem 4.3 (Brosamler's Theorem, [3])

Let O be an opened set of $\mathbb{R}^n$, let (X_t) be a bownian motion in $\mathbb{R}^n$, let ξ the meeting time in the complementary of O. Let f be a locally summable function in O, in which the laplacian in distributions term is a measure :

$$\frac{1}{2}\Delta f = \mu\ \ .$$

As to modify in a null measure set; f derivatives in distributions term are locally summable functions in O.

$$D^i f = j^i, i = 1,2,\dots,n$$

and we have $\int_0^t j^{i2}\ o\, X_s ds < \infty$.

$$For\ \ t < \xi\ ,\ \ f(X_t) = f(X_0) + \sum_1^n \int_0^t j^i(X_s)dx_s^i + A_t \qquad (10)$$

Where (A_t) is an adapted process defined on $[0,\xi[$, whose tragectories are null in O, continuous and locally with bounded variation on $[0,\xi[$, and it is the functional associated to μ.

In particular, if $\frac{1}{2}\Delta f$ is a locally summable function in O, we have :

$$\int_0^t |\Delta f o X_s| ds < \infty\ \ p.s\ \ (t < \xi)\ \ and$$

$$A_t = \frac{1}{2}\int_0^t \Delta f(X_s)ds.$$

V. CONCLUSION

Indeed, in this paper, we are essentially based on Brosamler's theorem, developed in [3] where the author works consisted to prove the existence of a quadratic variation of stochastic processes $f(x)$ in which X is a Brownian motion and f a harmonic function.

The equation (9), called Ito-Tanaka formula is an important result in the generalization of Ito formula in $\mathbb{R}$ set where the last part L_t^{α} is a continue crescent process called local time in α'' of the Brownian motion X; and in (10) that is the Ito formula generalized in $\mathbb{R}^n$ in which the last part A_t is a crescent process, whose trajectories are locally bounded in an $\mathbb{R}^n$ opened set, and so, the functional associated to the measure $\mu = \frac{1}{2}\Delta f$.

Perspective studies can use the generalized Ito formula, in the financial context of partial derivatives of Black-Scholes by bringing it back into spaces that can facilitate his resolution.

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