Fixed Point Theorems in Partial Fuzzy Metric Spaces

Kamal Wadhwa¹, Sanjay Choudhary², Hargovind Dubey³

Department of Mathematics, Govt. Narmada, PG College, Narmada Puram (M.P.)

Abstract:
The aim of this research article is basically to give some fixed point theorems in partial fuzzy metric space using a distance function.

Keywords: Partial metric, partial fuzzy metric, completeness, fixed point theorem.

1. Introduction


One of the generalizations of metric spaces is the notion of partial metric space which was given by Matthews [15] (1994) as an extension of metric space where the self-distance of any point is not necessarily equal to zero. This concept is motivated with the applications to computer science. Bukatin et al. [3] (2009) showed how the mathematics of nonzero self-distance for metric space has been established. They also considered some possible uses of partial metric spaces. Then, Valero [23](2005), Altun et al.[1] (2010), Haghi et al.[11] (2013) obtained some extensions of the result of Matthews [15] related to Banach fixed point theorem. In the last years, Yue and Gu [26] (2014), Sedghi et al. [22] (2015) and Gregori et al.[8] (2019) studied fuzzy partial metric spaces as a generalized of both partial metric space and fuzzy metric space.

In this work, we investigate some fixed point theorems in partial fuzzy metric space using a distance function.

2. We recall some basic definitions

Definition [21]. A partial metric space (shortly, PMS) on $X$ is a pair $(X,d)$ such that $X$ is a non-empty set and $d: X \times X \rightarrow \mathbb{R}^+ + \{0\}$ is a mapping providing the listed conditions for all $x, y, z \in X$:

(M1) $d((x,x)) \leq d(x,y)$,

(M2) $d(x,x) = d(x,y) = d(y,y)$ if and only if $x = y$,

(M3) $d(x,y) = d(y,x)$,

(M4) $d(x,z) \leq p(x,y) + p(y,z) - p(y,y)$ (Matthews [15] 1994).

Note that the self-distance of any point is not necessarily equal to zero in partial metric space. If $d(x,x) = 0$ for all $x \in X$, then the partial metric $d$ is an ordinary metric on $X$. So a partial metric is a generalization of ordinary metric (Matthews [15] 1994).
Definition [2.2]: A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions.

* is associative and commutative,
* is continuous,
$a * 1 = a$ for all $a \in [0, 1]$,
a * $b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Two typical examples of a continuous t-norm are $a * b = ab$ and $a * b = \min \{a, b\}$. (George and Veeramani [6] 1994).

Definition 2.3 [72]: A 3-tuple $(X, M, *)$ is called a fuzzy metric space if $X$ is an arbitrary (non-empty) set, $*$ is a continuous t-norm and $M$ is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and each $t, s > 0$,

$(M_1). M(x, y, t) > 0$,
$(M_2). M(x, y, t) = 1$ if and only if $x = y$,
$(M_3). M(x, y, t) = M(y, x, t)$,
$(M_4). M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
$(M_5). M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Example 2.4 [72]: Let $X$ be the set of all real numbers and $d$ be the Euclidean metric. Let $a * b = \min \{a, b\}$ for all $a, b \in [0,1]$. For each $t > 0$ and $x, y, \in X$,

Let $M(x, y, t) = \frac{t}{t+|x-y|}$. Then $(X, M, *)$ is a fuzzy metric space.

Proposition 1. If $(X, M, *)$ is a FMS, then $(M(x, y, .)) : ((0, \infty)) \to [[0,1]]$ is non-decreasing for all $x, y \in X$ (George and Veeramani [6] 1994).

Definition 2.5 [72]: A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to a point $x \in X$ if $\lim_{n \to \infty} M(x_n, x, t) = 1$ for any $t > 0$. The sequence $\{x_n\}$ is said to be Cauchy if $\lim_{n,m \to \infty} M(x_n, x_m, t) = 1$. The space $(X, M, *)$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$. (George and Veeramani [6] 1994).

Partial fuzzy metric space was defined by Sedghi et al.[22] (2015) as a generalization of partial metric and fuzzy metric spaces:

Definition 2.6. Let $X$ be a nonempty set. * be a continuous t-norm and
$M : X \times X \times ((0, \infty)) \to [0,1]$ be a mapping. If the listed conditions are satisfied for all $x, y, z \in X$ and $t, s > 0$, then the triplet $(X, M, *)$ is said to be a partial fuzzy metric space (shortly, PFMS) :

$$(PFM1) x = y \text{ if and only if } M(x, y, t) = M(x, x, t) = M(y, y, t)$$

$$(PFM2) M(x, x, t) \geq M(x, y, t) > 0,$$

$$(PFM3) M(x, y, t) = M(y, x, t),$$

$$(PFM4) M(x, z, t) * M(z, y, s) \leq M(x, y, \max\{t, s\}) * M(z, z, \max\{t, s\}),$$
\((PFM5)\) \(M(x, y, \ldots)\) is continuous on \(((0, \infty))\) (Sedghi et al., 2015).

**Remark 1.** Let \((X, M, \ast)\) be a PFMS.

1. If \(M(x, y, t) = 1\), then \(x = y\) from the conditions (PFM1) and (PFM2). But the converse of this implication need not be necessarily true. i.e., \(M(x, y, t)\) may not be equal to 1 whenever \(x = y\).
2. It is clear that \(M(x, z, t) \ast M(z, y, t) \leq M(x, y, t) \ast M(z, z, t)\) for all \(x, y, z \in X\) and \(t > 0\) from the conditions (PFM4) (Sedghi et al., 2015).

Note that each non-Archimedean FMS is a PFMS, but the converse implication may not be true.

**Example 2.** Let \((X, d)\) be a PMS and \(a \ast b = ab\) for all \(a, b \in [0, 1]\). Consider the mapping

\[M_d(x, y, t) = \frac{t}{t + d(x, y)}\]

Then \((X, M_d, \ast)\) is a PFMS which is called the standard PFMS. Note that \((X, M_d, \ast)\) is not a FMS (Sedghi et al., 2015).

There are some differences between PFMS and FMS. One of them, in a FMS \((X, M, \ast)\),

\[M((x, y, \ldots)) : ((0, \infty)) \rightarrow [[0, 1]]\]

is non-decreasing for all \(x, y \in X\), but in a PFMS \((X, M, \ast)\),

\[(M(x, y, \ldots)) : ((0, \infty)) \rightarrow [[0, 1]]\]

may not be non-decreasing function for all \(x, y \in X\).

**Proposition 2.** Let \(((X, M, \ast), ((b)\text{ a PFMS. If } b \geq c \text{ whenever } a \ast b \geq a \ast c \text{ for all } a, b, c \in [0, 1], then } M(x, y, \ldots)) : ((0, \infty)) \rightarrow [[0, 1]]\) is non-decreasing function for all \(x, y \in X\).

**Definition 2.7:** Let \((X, M, \ast)\) be a PFMS and \((x_n)\) be a sequence in \(X\).

1. \((x_n)\) is said to converge to a point \(x \in X\) if \(\lim_{n \rightarrow \infty} M(x_n, x, t) = M(x, x, t)\) for all \(t > 0\) and \(n \rightarrow \infty\) (Sedghi et al., 2015).

is said to be a Cauchy sequence if \(\lim_{n, m \rightarrow \infty} (x_n, x_m, t)\) exist for all \(t > 0\).

If \(\lim_{n, m \rightarrow \infty} (x_n, x_m, t) = 1\) then \((x_n)\) is called a \(1\)-Cauchy sequence.

2. If each Cauchy sequence (resp. \(1\)-Cauchy) \((x_n)\) converges to a point \(x \in X\) such that \(\lim_{n, m \rightarrow \infty} (x_n, x_m, t) = M(x, x, t)\), then \((X, M, \ast)\) is said to be complete (resp. \(1\)-complete).
Clearly, every 1-Cauchy sequence \((x_n)\) in \((X, M, \ast)\) is also a Cauchy sequence and every complete PFMS is a 1-complete space.

Proposition 3. Let \((X, M, \ast)\) be a PFMS and \((x_n)\) be a sequence in \(X\) such that
\[
\lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(x_n, x, t) = M(x, x, t).
\]
If \(b \geq c\) whenever \(a \ast b \geq a \ast c\) for all \(a, b, c \in [0, 1]\). Then
\[
\lim_{n \to \infty} M(x_n, y, t) = M(x, y, t) \text{ for all } y \in X \text{ and } t > 0 \quad (\text{Sedghi et al.}[22] 2015).
\]

We define a continuous function \(\psi : [0,1] \to [0,1]\) satisfying the following conditions:

(i) \(\psi\) is nondecreasing on \([0,1]\),
(ii) \(\psi(t) > t\) for each \(t \in (0,1)\).

We note that \(\psi(1) = 1 \& \psi(t) \geq t\) for all \(t \in [0,1]\).

**Theorem 1:** Let \((X, M, \ast)\) be a complete PFMS such that \(\lim_{n \to \infty} M(x, y, t) = 1 \forall x, y \in X\) and \(f: X \to X\) be a self-map such that for all \(x, y \in X\) and \(k \in (0,1)\)
\[
M(fx, fy, kt) \geq \emptyset \{M(x, f(x), t)\} \quad (1)\text{hold, then there exist a unique fixed point for } f \text{ in } X.
\]

Proof: For each \(x_0 \in X\) and \(n \in N\), put \(x_{n+1} = fx_n\)

It follows From (1) that
\[
M(x_n, x_{n+1}, kt) = M(fx_{n-1}, fx_n, kt) \geq \emptyset \{M(x_{n-1}, f x_{n-1}, t)\} = \emptyset \{M(x_{n-1}, x_n, t)\} \geq M(x_{n-1}, x_n, t)
\]

Then, we have
\[
M(x_n, x_{n+1}, t) \geq M \left( x_{n-1}, x_n, \frac{t}{k^n} \right) \geq M \left( x_{n-2}, x_{n-1}, \frac{t}{k^{n-1}} \right) \ldots \geq M \left( x_0, x_1, \frac{t}{k^n} \right) \text{ for all } t > 0.
\]

Taking \(\lim n \to \infty\), we get
\[
\lim_{n \to \infty} M(x_n, x_{n+1}, t) = 1, \quad \forall t > 0
\]

Let \(n, m \in N\) and we may assume that \(n < m\).
\[
M(x_n, x_m, t) \geq M(x_n, x_{n+1}, t) \ast M(x_{n+1}, x_{n+2}, t) \geq M(x_n, x_{n+1}, t) \ast M(x_{n+1}, x_{n+2}, t)
\]
\[
\geq M(x_n, x_{n+1}, t) \ast M(x_{n+1}, x_{n+2}, t) \ast M(x_{n+2}, x_{m}, t)
\]
\[
\geq \cdots \geq M(x_n, x_{n+1}, t) \ast M(x_{n+1}, x_{n+2}, t) \ast \ldots \ast M(x_{m-1}, x_m, t)
\]

Thus, we obtain
\[
\lim_{n, m \to \infty} M(x_n, x_m, t) = 1, \forall t > 0
\]

Hence \(\{x_n\}\) is a Cauchy sequence in \((X, M, \ast)\).

Since \((X, M, \ast)\) is a complete PFMS, there exists a point \(x \in X\) such that \(\{x_n\}\) converges to \(x\). Besides,
\[
\lim_{n \to \infty} M(x_n, x, t) = M(x, x, t) = \lim_{n \to \infty} (x_n, x, t) = 1 \forall t > 0.
\]

Then,
\[
M(f(x), x, t) \geq M(f(x), x, t) \ast M(x_n, x, t) \geq M(f(x), x, t) \ast M(x_n, x, t)
\]
\[
\geq M(f(x), f(x_{n-1}), t) \ast M(x, x, t)
\]
\[
\geq M(x, x_{n-1}, t) \ast M(x, x, t)
\]

Therefore, we have \(M(f(x), x, t) = 1\). This means that \(f(x) = x\).

Hence \(x\) is a fixed point of \(f\).

Now, we show that \(x\) is a unique fixed point of \(f\). Assume that \(x \neq y\). Then we get
\[
M(x, y, t) = M(f(x), y, t) \geq \emptyset (M(f(x), f(y), t)) = \emptyset (M(x, y, t)) > M(x, y, t)
\]

Which is a contradiction. Hence, we have \(x = y\).

**Theorem 2:** Let \((X, M, \ast)\) be a complete PFMS such that \(\lim_{n \to \infty} M(x, y, t) = 1 \forall x, y \in X\) and \(f, g: X \to X\) be two self-maps such that for all \(x, y \in X\)
\[
M(fx, fy, kt) \geq \emptyset \{M(gx, gy, t)\} \quad (1)\text{hold, then there exist a unique common fixed point for } f \& g \text{ in } X \text{ where } k \in (0,1).
\]

Proof: For each \(x_0 \in X\) and \(n \in N\), put \(x_{n+1} = fx_n\), and \(x_{n+2} = gx_{n+1} \)
We have,
\[
M(x_{n+1}, x_{n+2}, kt) = M(fx_n.fx_{n+1}, kt) \geq \varnothing(M(gx_n.gx_{n+1}, t)) \\
\geq M(gx_n.gx_{n+1}, t) = M(x_{n+1}, x_{n+2}, t)
\]
\[
M(x_{n+1}, x_{n+2}, t) \geq M \left( x_{n+1}, x_{n+2}, \frac{t}{k^n} \right) \geq \cdots \geq M \left( x_{n+1}, x_{n+2}, \frac{t}{k^n} \right)
\]
Taking \( \lim n \to \infty \), we get
\[
\lim_{n\to\infty} M(x_{n+1}, x_{n+2}, t) = 1, \quad \forall t > 0
\]
Let \( n, m \in N \) and we may assume that \( n < m \).
\[
M(x_n, x_m, t) \geq M(x_n, x_n, t) * M(x_n, x_n, t) \geq M(x_n, x_n, t) * M(x_n, x_n, t)
\]
\[
\geq M(x_n, x_n, t) * M(x_n, x_n, t) * M(x_n, x_n, t) \geq M(x_n, x_n, t) \geq \cdots \geq M(x_n, x_n, t) * M(x_n, x_n, t) * \cdots * M(x_n, x_n, t)
\]
Thus, we obtain
\[
\lim_{n,m\to\infty} M(x_n, x_m, t) = 1, \forall t > 0
\]
Hence \( \{x_n\} \) is a Cauchy sequence in \( (X, M, *) \)
Since \( (X, M, *) \) is a complete PFMS, there exists a point \( x \in X \) such that \( \{x_n\} \) converges to \( x \). Besides,
\[
\lim_{n\to\infty} M(x_n, x, t) = M(x, x, t) = \lim_{n,m\to\infty} (x_n, x_m, t) = 1 \forall t > 0
\]
Then,
\[
M(f(x), x, t) \geq M(f(x), x, t) * M(x_n, x_n, t) \geq M(f(x), x_n, t) \geq M(f(x), f(x_n-1), t) * M(x, x, t)
\]
\[
\geq M(x, x_n-1, t) * M(x, x, t)
\]
Therefore, we have \( M(f(x), x, t) = 1 \). This means that \( f(x) = x \).
Similarly we can show \( g(x) = x \)
Hence \( x \) is a common fixed point of \( f \& g \).
Now, we show that \( x \) is a unique fixed point of \( f \& g \). Assume that \( x \neq y \). Then we get
\[
M(x, y, t) = M(f(x, f(y), t) \geq \varnothing(M(gx, gy, t)) = \varnothing(M(x, y, t)) > M(x, y, t)
\]
Which is a contradiction. Hence, we have \( x = y \)

REFERENCES