Fixed Point Theorems in Partial Fuzzy Metric Spaces

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Abstract:
The aim of this research article is basically to give some fixed point theorems in partial fuzzy metric space using a distance function.

Keywords: Partial metric, partial fuzzy metric, completeness, fixed point theorem.

1.Introduction

The theory of fuzzy set initiated by Zadeh[27] in 1965 to understand the uncertainty using the membership degree. This theory has found many applications in a lot of fields since there are many real-life problems where the nature of a given system possesses fuzziness. The concept of fuzzy metric space was defined in two different ways by Kramosil and Michalek [12] (1975) and Kaleva and Seikkala [13] (1984). Later, George and Veeramani [6] (1994) redefined the notion of fuzzy metric space in a slightly different way from Kramosil and Michalek [12]to construct a Hausdorff (T_2) topology from a given fuzzy metric space. Grabiec [10] (1988) gave the Banach contraction theorem in the fuzzy metric setting in the means of Kramosil and Michalek [12] (1975). Afterward, many authors (Piera 2001[20], Vasuki [25] 2003, Mihet [16] 2004/2008, Rodriguez-Lopez and Romaguera [21]2004, Gregori et al.[7] 2010) proved some fixed point theorems on fuzzy metric space in several senses and became interested in topological properties of fuzzy metric spaces.
One of the generalizations of metric spaces is the notion of partial metric space which was given by Matthews [15] (1994) as an extension of metric space where the self-distance of any point is not necessarily equal to zero. This concept is motivated with the applications to computer science. Bukatin et al. [3] (2009) showed how the mathematics of nonzero self-distance for metric space has been established. They also considered some possible uses of partial metric spaces. Then, Valero [23](2005), Altun et al.[1] (2010), Haghi et al.[11] (2013) obtained some extensions of the result of Matthews [15] related to Banach fixed point theorem. In the last years, Yue and Gu [26] (2014), Sedghi et al. [22] (2015) and Gregori et al.[8] (2019) studied fuzzy partial metric spaces as a generalized of both partial metric space and fuzzy metric space.

In this work, we investigate some fixed point theorems in partial fuzzy metric space using a distance function.

2.We recall some basic definitions

Definition [2.1]. A partial metric space (shortly, PMS) on X is a pair (X, d) such that X is a non-empty set and $\mathrm{d}: \mathrm{X} \times \mathrm{X} \rightarrow \mathrm{R}^{+}$ is a mapping providing the listed conditions for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{X}$:

$$(\mathrm{M1})\ \mathrm{d}((\mathrm{x}, \mathrm{x})) \leq \mathrm{d}(\mathrm{x}, \mathrm{y}),$$

$$(\mathrm{M2})\mathrm{d}(\mathrm{x}, \mathrm{x}) = \mathrm{d}(\mathrm{x}, \mathrm{y}) = \mathrm{d}(\mathrm{y}, \mathrm{y}) \text{ if and only if } \mathrm{x} = \mathrm{y},$$

$$(\mathrm{M3})\ \mathrm{d}(\mathrm{x}, \mathrm{y}) = \mathrm{d}(\mathrm{y}, \mathrm{x}),$$

$$(\mathrm{M4})\ \mathrm{d}(\mathrm{x}, \mathrm{z}) \leq \mathrm{p}(\mathrm{x}, \mathrm{y}) + \mathrm{p}(\mathrm{y}, \mathrm{z}) - \mathrm{p}(\mathrm{y}, \mathrm{y}) \text{ (Matthews [15] 1994).}$$

Note that the self-distance of any point is not necessarily equal to zero in partial metric space. If $\mathrm{d}(\mathrm{x}, \mathrm{x}) = 0$ for all $\mathrm{x} \in \mathrm{X}$, then the partial metric d is an ordinary metric on X. So a partial metric is a generalization of ordinary metric (Matthews [15] 1994).

Definition [2.2]: A binary operation$*$: $[0,1] \times [0,1] \to [0,1]$ is a continuous t - norm if it satisfies the following conditions.

$*$ is associative and commutative,
$*$ is continuous,
$a * 1 = a$ for all $a \in [0,1]$,
$a * b \le c * d$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of a continuous t – norm are $a * b = ab$ and $a * b = \min\{a, b\}$.
(George and Veeramani [6] 1994).

Definition 2.3 [72]: A **3-** tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non – empty) set, $*$ is a continuous t – norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and each $t, s > 0$,

(M_1). $M(x, y, t) > 0$,

(M_2). $M(x, y, t) = 1$ if and only if $x = y$,

(M_3). $M(x, y, t) = M(y, x, t)$,

(M_4). $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(M_5). $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Example 2.4 [72]: Let X be the set of all real numbers and d be the Euclidean metric. Let $a * b = \min\{a, b\}$ for all $a, b \in [0,1]$. For each $t > 0$ and $x, y, \in X$,

Let $M(x, y, t) = \frac{t}{t + d(x,y)}$. Then $(X, M, *)$ is a fuzzy metric space.

Proposition 1. If $(X, M, *)$ is a FMS, then $(M(x, y, \cdot)) : ((0, \infty)) \to [[0,1]]$ is non-decreasing for all $x, y \in X$ (George and Veeramani [6] 1994).

Definition 2.5 [72]: A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for any $t > 0$. The sequence $\{x_n\}$ is said to be Cauchy if $\lim_{n,m\to\infty} M(x_n, x_m, t) = 1$**.** The space $(X, M, *)$ is said to be complete if every Cauchy sequence in X is convergent in X. (George and Veeramani [6] 1994).

Partial fuzzy metric space was defined by Sedghi et al.[22] (2015) as a generalization of partial metric and fuzzy metric spaces:

Definition 2.6. Let X be a nonempty set, $*$ be a continuous t-norm and $M : X \times X \times ((0, \infty)) \to [0,1]$ be a mapping. If the listed conditions are satisfied for all $x, y, z \in X$ and $t, s > 0$, then the triplet $(X, M, *))$ is said to be a partial fuzzy metric space (shortly, PFMS) :

$$(PFM1)\ x = y \text{ if and only if } M(x, y, t) = M(x, x, t) = M(y, y, t)$$

$(PFM2) M(x, x, t) \ge M(x, y, t) > 0$,

$(PFM3) M(x, y, t) = M(y, x, t)$,

$(PFM4) M(x, z, t) * M(z, y, s) \le M(x, y, max\{t, s\}) * M(z, z, max\{t, s\})$,

$(PFM5) M(x, y, .)$ Is continuous on $(((0, \infty))$(Sedghi et al., 2015).

Remark 1. Let $(X, M, *)$ be a PFMS.

(1) If $M(x, y, t)) = 1$, then $x = y$ from the conditions (PFM1) and (PFM2). But the converse of this implication need not be necessarily true. i.e., $M(x, y, t)$ may not be equal to 1 whenever $x = y$.

(2) It is clear that $M(x, z, t) * M(z, y, t) \leq M(x, y, t) * M(z, z, t)$ for all $x, y, z \in X$ and $t > 0$ from the conditions (PFM4) (Sedghi et al.[22] 2015).

Note that each non-Archimedean FMS is a PFMS, but the converse implication may not be true.

Example 2. Let $((X, d))$ be a PMS and $a * b = ab$ for all $a, b \in [0,1]$. Consider the mapping $M_d : X \times X \times (0, \infty) \to [[0, 1]]$ $defined$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, Md, *)$ is a PFMS which is called the standard PFMS. Note that $(X, Md, *)$ is not a FMS (Sedghi et al.[22] 2015).

There are some difference between PFMS and FMS. One of them, in a FMS $(X, M, *)$, $\mathcal{M}((x, y, .)) : ((0, \infty)) \to [[0,1]]$ is non-decreasing function for all $x, y \in X$, but in a PFMS $(X, M, *)$, $(M(x, y, .)) : ((0, \infty)) \to [[0,1]]$ may not be non-decreasing function for all $x, y \in X$.

Proposition 2. Let $((((X, M, *)(((b$e a PFMS. If $b \geq c$ whenever $a * b \geq a * c$ for all $a, b, c \in [0, 1]$, then $M(x, y, .)) : ((0, \infty)) \to [[0,1]$ is non-decreasing function for all $x, y \in U$. (Sedghi et al.[22] 2015).

Definition 2.7: Let $(X, M, *)$ be a PFMS and (x_n) be a sequence in X.

(i) (x_n) is said to converge to a point $x \in X$ if $lim\ M(x_n, x, t) = M(x, x, t)$ for all $t > 0$ and $n \to \infty$.(Sedghi et al.[22] 2015).

is said to be a Cauchy Sequence if $\lim_{n,m \to \infty} (x_n, x_m, t)$ exist for all $t > 0$.

If $\lim_{n,m \to \infty} (x_n, x_m, t) = 1$ then (x_n) is called a 1- Cauchy sequence.

(ii)

If each Cauchy sequence (resp. 1- Cauchy) (x_n) converges to a point $x \in X$ such that $\lim_{n,m \to \infty} (x_n, x_m, t) = M(x, x, t)$, then $(X, M, *)$ is said to be complete (resp.1-complete)

Clearly, every 1-Cauchy sequence (x_n) in $(X, M,*)$ is also a Cauchy sequence and every complete PFMS is a 1-complete space.

Proposition 3. Let $(X, M,*)$ be a PFMS and (x_n) be a sequence in X such that $\lim_{n\to\infty} M(x_n, x_n, t) = \lim_{n\to\infty} (x_n, x, t) = M(x, x, t)$. If $b \geq c$ whenever $a * b \geq a * c$ for all $a, b, c \in [0,1]$.Then $\lim_{n\to\infty} M(x_n, \gamma, t) = M(x, \gamma, t)$ for all $\gamma \in X$ and $t > 0$ (Sedghi et al.[22] 2015).

We define a continuous function $\psi : [0,1] \to [0,1]$ satisfying the following conditions:

(i) ψ is nondecresing on $[0,1]$,

$\psi(t) > t$ for each $t \in (0,1)$

We note that $\psi(1) = 1 \& \psi(t) \geq t$ for all t in $[0,1]$.

Theorem1: let $(X, M,*)$ be a complete PFMS such that $\lim_{n\to\infty} M(x, y, t) = 1\ \forall x, y \in X$,and $f: X \to X$ be a self map such that for all $x, y \in X$ and $k \in (0,1)$

$M(fx, fy, kt) \geq \emptyset\{M(x, fx, t)\}$ … (1)hold, then there exist a unique fixed point for f in X.

Proof: For each $x_0 \in X$ and $n \in N$, put $x_{n+1} = fx_n$

It follows From (1) that

$$M(x_n, x_{n+1}, kt) = M(fx_{n-1}, fx_n, kt) \geq \emptyset\{M(x_{n-1}, fx_{n-1,}t)\} = \emptyset\{M(x_{n-1}, x_{n,}\ t)\} \geq M(x_{n-1}, x_{n,}\ t)$$

Then, we have

$$M(x_n, x_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{t}{k}\right) \geq M\left(x_{n-2}, x_{n-1}, \frac{t}{k^2}\right) \ldots . \geq M(x_0, x_1, \frac{t}{k^n}) \text{ for all } t > 0.$$

Taking $lim\ n \to \infty$, we get

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1, \qquad \forall t > 0$$

Let $n, m \in N$ and we may assume that $n < m$.

$$\begin{aligned} M(x_n, x_m, t) &\geq M(x_n, x_m, t) * M(x_{n+1}, x_{n+1}, t) \geq M(x_n, x_{n+1}, t) * M(x_m, x_{n+1}, t) \\ &\geq M(x_n, x_{n+1}, t) * M(x_m, x_{n+1}, t) * M(x_{n+2}, x_{n+2}, t) \\ &\geq M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t) * M(x_{n+2}, x_m, t) \\ &\geq \cdots \geq M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t) * \ldots * M(x_{m-1}, x_m, t) \end{aligned}$$

Thus, we obtain

$$\lim_{n,m\to\infty} M(x_n, x_m, t) = 1, \forall\ t > 0$$

Hence $\{x_n\}$ is a Cauchy sequence in $(X, M,*)$

Since $(X, M,*)$ is a complete PFMS, there exists a point $x \in X$ such that $\{x_n\}$ converges to x.Besides, $\lim_{n\to\infty} M(x_n, x, t) = M(x, x, t) = \lim_{n,m\to\infty} (x_n, x_m, t) = 1\ \forall\ t > 0$. Then,

$$\begin{aligned} M(f(x), x, t) &\geq M(f(x), x, t) * M(x_n, x_n, t) \geq M(f(x), x_n, t\ ) * M(x_n, x, t) \\ &\geq M(f(x), f(x_{n-1}), t) * M(x, x, t) \\ &\geq M(x, x_{n-1,}t) * M(x, x, t) \end{aligned}$$

Therefore, we have $M(f(x), x, t) = 1$. This means that $f(x) = x$.

Hence x is a fixed point of f.

Now, we show that x is a unique fixed point of f. Assume that $x \neq y. Then\ we\ get$

$$M(x, y, t) = M(fx, fy, t) \geq \emptyset(M(f(x), f(y), t)) = \emptyset(M(x, y, t)) > M(x, y, t)$$

Which is a contradiction. Hence , we have $x = y$

Theorem2: let $(X, M,*)$ be a complete PFMS such that $\lim_{n\to\infty} M(x, y, t) = 1\ \forall x, y \in X$ and $f, g: X \to X$ be two self maps such that for all $x, y \in X$

$M(fx, fy, kt) \geq \emptyset\{M(gx, gy, t)\}$ … (1)hold, then there exist a unique common fixed point for $f \&\ g$ in X $where$ $k \in (0,1)$.

Proof: For each $x_0 \in X$ and $n \in N$, put $x_{n+1} = fx_n$, and $x_{n+2} = gx_{n+1}$

We have,

$$M(x_{n+1}, x_{n+2}, kt) = M\big(fx_{n,}fx_{n+1,}kt\big) \geq \emptyset(M\big(gx_{n,}gx_{n+1}, t\big))$$
$$\geq M\big(gx_{n,}gx_{n+1}, t\big)\big) = M(x_{n+1}, x_{n+2}, t)$$

$$M(x_{n+1}, x_{n+2}, t) \geq M\left(x_{n+1,}x_{n+2,}\frac{t}{k}\right) \geq \cdots \geq M\left(x_{n+1,}x_{n+2,}\frac{t}{k^n}\right)$$

Taking $lim\ n \to \infty$, we get

$$\lim_{n\to\infty} M(x_{n+1}, x_{n+2}, t) = 1, \qquad \forall t > 0$$

Let $n, m \in N$ and we may assume that $n < m$.

$$M(x_n, x_m, t) \geq M(x_n, x_m, t) * M(x_{n+1}, x_{n+1}, t) \geq M(x_n, x_{n+1}, t) * M(x_m, x_{n+1}, t)$$
$$\geq M(x_n, x_{n+1}, t) * M(x_m, x_{n+1}, t) * M(x_{n+2}, x_{n+2}, t)$$
$$\geq M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t) * M(x_{n+2}, x_m, t)$$
$$\geq \cdots \geq M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t) * \ldots * M(x_{m-1}, x_m, t)$$

Thus, we obtain

$$\lim_{n,m\to\infty} M(x_n, x_m, t) = 1, \forall\, t > 0$$

Hence $\{x_n\}$ is a Cauchy sequence in $(X, \mathrm{M}, *)$

Since $(X, M, *)$ is a complete PFMS, there exists a point $x \in X$ such that $\{x_n\}$ converges to x.Besides, $\lim_{n\to\infty} M(x_n, x, t) = M(x, x, t) = \lim_{n,m\to\infty} (x_n, x_m, t) = 1\ \forall\, t > 0$. Then,

$$M(f(x), x, t) \geq M(f(x), x, t) * M(x_n, x_n, t) \geq M(f(x), x_n, t\ ) * M(x_n, x, t)$$
$$\geq M(f(x), f(x_{n-1}), t) * M(x, x, t)$$
$$\geq M\big(x, x_{n-1,}t\big) * M(x, x, t)$$

Therefore, we have $M(f(x), x, t) = 1$. This means that $f(\mathrm{x}) = x$.

Similarly we can show $g(x) = x$

Hence x is a common fixed point of $f\&g$.

Now, we show that x is a unique fixed point of $f\&g$. Assume that $x \neq y. Then\ we\ get$

$$M(x, y, t) = M(fx, fy, t) \geq \emptyset\big(M(gx, gy, t)\big) = \emptyset(M(x, y, t)) > M(x, y, t)$$

Which is a contradiction. Hence , we have $x = y$

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