n^{th} Power (n Is A Real Number) of A Square Matrix by Using Diagonalization Method, Applications of Diagonalization

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Abstract:

The aim of the present paper is tofind thenth root of a square matrix by the process of diagonalization. We can find the nth root of a square matrix if it is diagonalizable. A diagonal matrix is a square matrix whose elements other than the principal diagonal are zero. A square matrix is said to be diagonalizable if there exists an invertible square matrix P such that
$A=PDP^{-1}$ or AP=PD, where D is a Diagonal matrix. The process of diagonalizing a matrix is defined as diagonalization. For the process of diagonalizing a matrix, we have to find eigen values, eigen vectors of the matrix.

Key words: Square matrix, Diagonal matrix, Eigen values, Eigen vectors.

1. INTRODUCTION

(1) Matrix: An arrangement of real or complex numbers in rows and columns is defined as a Matrix. An example of a matrix is $\begin{pmatrix} 3 & 0 & 1 \\ 1 & -4 & 7 \end{pmatrix}$. It contains 2 rows, three columns. We define its order as 2x3.We also defines it **as rectangular matrix** as the no. of rows is not equal to no. of columns.

(2) Square matrix: A Matrix in which the no. of rows is equal to no. of columns is defined as a square matrix. An example of a square matrix is $\begin{pmatrix} 2 & 4 & 7 \\ 6 & 10 & 9 \\ 1 & 0 & 8 \end{pmatrix}$.I t contains 3 rows and 3 columns. Hence its order is 3x3.

(3) Principal diagonal of a square matrix: The straight line joiningthe element of first row**,** first column, the element of second row, second column, ------------, the nth row, nth column is defined as the principal diagonal of the matrix.

(4) Diagonal matrix: A square matrix whose elements other than the principal diagonal are zero is defined to be diagonalizable matrix.

(5) Identity matrix: A square matrix in which the principal diagonal elements are 1 and all the other elements are zeroes is defined as Identity matrix.$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$is an example of an identity matrix of order 3x3.An identity matrix of order 3x3 is denoted by I_{3X3}.

(6) Determinant of a 2x2 square matrix: The determinant of a 2x2 square matrix$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$is defined as **ad-bc.** The determinant of a square matrix A is denoted by $|A|\ or\ \det(A)$.

(7) Minor of an element of a 3x3square matrix: Minor of an element of a square matrix is defined as the determinant of 2x2 matrix obtained after removing the row, column along it. The minor of an element a_{ij} (the element along ith row, jth column) is denoted by M_{ij}.

EX: Consider the matrix A=$\begin{pmatrix} 3 & 1 & -2 \\ 0 & 7 & 8 \\ -4 & 9 & 6 \end{pmatrix}$.Minor of 3 is $\begin{vmatrix} 7 & 8 \\ 9 & 6 \end{vmatrix}$=7(6)-8(9) =42-72=-30

Minor of 8 is $\begin{vmatrix} 3 & 1 \\ -4 & 9 \end{vmatrix}$=3(9) -(-4) =27+4=31

(8) Cofactor of an element of a 3x3 matrix: The cofactor of an element aij(the element along ith row, jth column) in a matrix is defined as $(-1)^{i+j}\ M_{ij}$., Here M_{ij} is minor of a_{ij}.

EX: Consider the matrix $A = \begin{pmatrix} 0 & -10 & 14 \\ 5 & 7 & 2 \\ 11 & 20 & 50 \end{pmatrix}$.The cofactor of 0 is $(-1)^{1+1}\begin{vmatrix} 7 & 2 \\ 20 & 50 \end{vmatrix}$=

$(-1)^2$[7(50)-2(20)] =350-40=310.The cofactor of 5 is $(-)^{2+1}\begin{vmatrix} -10 & 14 \\ 20 & 50 \end{vmatrix}$= $(-1)^3$[(-10)50-14(20)]

=-(-500-280) =-(-780) =780.

(9) Determinant of a Square Matrix: Given any square matrix, the sum of product of elements of any row or column with their corresponding cofactors is defined as the determinant of the matrix.

EX: Consider the matrix A=$\begin{pmatrix} -6 & 5 & 3 \\ 8 & -2 & 4 \\ 7 & 8 & 1 \end{pmatrix}$.To find the determinant of this matrix, we have to select any row or any column and we have to take the sum of the product of the elements of the row or column with their cofactors.

Let us take the elements 5, -2,8 of second column.

Co factor of 5 is $(-1)^{1+2}\begin{vmatrix} 8 & 4 \\ 7 & 1 \end{vmatrix}$ = (-1) [8-28] =20.

Cofactor of -2 is $(-1)^{2+2}\begin{vmatrix} -6 & 3 \\ 7 & 1 \end{vmatrix}$= (1) [-6-21] =-27.

Cofactor of 8 is $(-1)^{3+2}\begin{vmatrix} -6 & 3 \\ 8 & 4 \end{vmatrix} = (-1)$[-24-24] =48.

The sum of product of elements 5, -2,8 of second column with their cofactors =5(20) +

(-2) (-27) +8(48) =100+54+384=538.Hence det(A)=538.

(10) Non-Singular Matrix: A Square matrix whose determinant is not zero is defined as a non-singular matrix. A Square matrix whose determinant is not zero is defined as a Singular matrix.

Ex: Consider the matrix A=$\begin{pmatrix} 2 & 8 \\ 7 & -3 \end{pmatrix}$.Its determinant is 2(-3)-8(7) =-6-56=-62

Det(A)=-62⇒det(A)≠0. Hence the matrix A=$\begin{pmatrix} 2 & 8 \\ 7 & -3 \end{pmatrix}$ is anon-singular matrix.

(11) Multiplication of two matrices: Given any two matrices, we can multiply the matrices if the number of columns of first matrix is equal to the number of rows of second matrix.

To multiply two matrices, we multiply each element of a row of first matrix with each element of its corresponding column. It is illustrated below.

Consider two matrices A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,B=$\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ of order 2x2

The multiplication of A, B is AB=$\begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$

Consider two matrices A=$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$,B=$\begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix}$

The multiplication of A, B is AB=$\begin{pmatrix} ap+bs+cv & aq+bt+cw & ar+bu+cx \\ dp+es+fv & dq+et+fw & dr+eu+fx \\ gp+hs+iv & gq+ht+iw & gr+hu+ix \end{pmatrix}$

(12) Inverse of a Square matrix: If for a square matrix A, there exists a square matrix B such that AB=BA=I. Where I is an identity matrix then B is defined as inverse of A and A is defined as inverse of A. The inverse of A is denoted by A^{-1}.

(13) Cofactor matrix ofa square matrix: The Square matrix obtained by replacing its elements with their cofactors is defined as a cofactor matrix.

(14) Transpose of a matrix: The matrix obtained by interchanging rows in to columns or columns in to rows of a square matrix is defined as the Transpose of the matrix.

Ex: Consider a matrix A= ($\begin{matrix} 7 & -9 & 1 \\ 0 & 2 & 10 \end{matrix}$). By interchanging rows in to columns we get the transpose matrix of A, it is A^T=($\begin{matrix} 7 & 0 \\ -9 & 2 \\ 1 & 10 \end{matrix}$)

(15) Adjoint matrix of a square matrix: The transpose of cofactor matrix of a given square matrix isdefined as the Adjoint matrix of a given square matrix. The Adjoint matrix of a square matrix A is denoted by adj(A).

EX: Consider the matrix $\begin{pmatrix} \mathbf{2} & \mathbf{0} & \mathbf{-6} \\ \mathbf{5} & \mathbf{8} & \mathbf{4} \\ \mathbf{1} & \mathbf{-2} & \mathbf{9} \end{pmatrix}$**.**The Cofactor of 2 is $(-1)^{1+1}\begin{vmatrix} 8 & 4 \\ -2 & 9 \end{vmatrix}$=

$(-1)^2$[8(9)-4(-2)] = (1) [72+8] =80

Similarly, the cofactor of 0 is -41, cofactor of -6 is -18, cofactor of 5 is -12, cofactor of 8 is 24, cofactor of 4 is 4, cofactor of 1 is 48, cofactor of -2 is -38, cofactor of 9 is 16.

Hence cofactor of matrix of A= $\begin{pmatrix} 80 & -41 & -18 \\ -12 & 24 & 4 \\ 48 & -38 & 16 \end{pmatrix}$

The adjoint matrix of A is Adj(A)= $\begin{pmatrix} 80 & -12 & 48 \\ -41 & 24 & -38 \\ -18 & 4 & 16 \end{pmatrix}$

Theorem: If Ais an nxn matrix then A(Adj(A)=Adj(A)(A) =Det(A). I where I is an nxn unit matrix.

(16) Formula for inverse of a squarematrix: By above theorem**,** If A is a non-singular square matrix, then inverse of A is $A^{-1}= \frac{adj(A)}{\det(A)}$.

(17) Eigen value, Eigen vector of a square matrix: For a given square matrix A, if there

Exists a non-zero vector X such that AX=λX, where λ is a real or complex number then X is defined as Eigen vector and λ is defined as eigen value.

We can find eigen value of a square matrix from AX=λX,

Characteristic Equation: AX=λX⇒AX=λIX⇒AX-λIX=O⇒(A-λI)X=O.To get a non-zero vector, the condition is is $|A - \lambda I|=0$.$|A - \lambda I|=0$is defined as the characteristic equation. Fromcharacteristic equation of a square matrix, we find eigen values.

After finding eigen values of the square matrix, we find eigen vectors by using the equation (A-λI)X=O.

Ex: Consider the matrix A= $\begin{pmatrix} 10 & 4 \\ 5 & 2 \end{pmatrix}$,consider a vector X= $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

AX= $\begin{pmatrix} 10 & 4 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.= $\begin{pmatrix} 18 \\ 9 \end{pmatrix}$=9$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.Hence AX=λX, where$\lambda = 9$

Hence $\lambda = 9$ is an eigen value and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is eiden vector corresponding to the eigen value of $\lambda = 9$.

EX: Find the eigen values of the matrix A= $\begin{pmatrix} 5 & 9 \\ 1 & 3 \end{pmatrix}$ by finding the characteristic equation of A.

The characteristic equation of A is given by.$|A - \lambda I|=0$

$A - \lambda I= \begin{pmatrix} 5 & 9 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 9 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 5-\lambda & 9 \\ 1 & 3-\lambda \end{pmatrix}$

$|A - \lambda I|=0 \Rightarrow \begin{vmatrix} 5-\lambda & 9 \\ 1 & 3-\lambda \end{vmatrix} =0 \Rightarrow (5-\lambda)(3-\lambda)-9(1) =0$

$\Rightarrow 15-5\lambda-3\lambda+\lambda^2 - 9=0 \Rightarrow 6-8\lambda+\lambda^2=0 \Rightarrow \lambda^2-8\lambda+6=0 \Rightarrow \lambda = \frac{-(-8)\pm\sqrt{(-8)^2-4(1)(6)}}{2(1)}$

$\Rightarrow \lambda = \frac{8\pm\sqrt{40}}{2} \Rightarrow \lambda = \frac{8+\sqrt{40}}{2}, \lambda = \frac{8-\sqrt{40}}{2}$

Hence the eigen values of the matrix $\begin{pmatrix} 5 & 9 \\ 1 & 3 \end{pmatrix}$ are $\frac{8+\sqrt{40}}{2}, \frac{8-\sqrt{40}}{2}$

We find eigen vectors by using (A-λI)X=O.

A= $\begin{pmatrix} 5 & 9 \\ 1 & 3 \end{pmatrix}$,$\Rightarrow \lambda = \frac{8+\sqrt{40}}{2}, \frac{8-\sqrt{40}}{2}$$\Rightarrow$(A-λ$I$)X=O gives

$$\begin{pmatrix} 5-\left(\frac{8+\sqrt{40}}{2}\right) & 9 \\ 1 & 3-\left(\frac{8+\sqrt{40}}{2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5-\left(\frac{8-\sqrt{40}}{2}\right) & 9 \\ 1 & 3-\left(\frac{8-\sqrt{40}}{2}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow\begin{pmatrix} \frac{2-\sqrt{40}}{2} & 9 \\ 1 & \frac{-2-\sqrt{40}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2+\sqrt{40}}{2} & 9 \\ 1 & \frac{-2+\sqrt{40}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow\left(\frac{2-\sqrt{40}}{2}\right)x+9y=0,\ x+\left(\frac{-2-\sqrt{40}}{2}\right)y=0,$

$\&\ \left(\frac{2+\sqrt{40}}{2}\right)x+9y=0,\ x+\left(\frac{-2+\sqrt{40}}{2}\right)y=0$

$\Rightarrow 2x-(2+\sqrt{40})y = 0, 2x+(-2-\sqrt{40})y = 0\ \&2x-(2-\sqrt{40})y = 0,$

$$2x+\left(-2+\sqrt{40}\right)y = 0;$$

$\Rightarrow 2x-(2+\sqrt{40})y = 0 \& 2x+(2+\sqrt{40})y = 0$

$\Rightarrow x-(1+\sqrt{10})y = 0 \& x+(1+\sqrt{10})y=0 \Rightarrow x= (1+\sqrt{10})y \&\ x=-(1+\sqrt{10})y$

By assuming y=1 in both the equations we have $x=1+\sqrt{10}$& $x=-(1+\sqrt{10})$

Hence vectors corresponding to eigen values $\frac{8+\sqrt{40}}{2}, \frac{8-\sqrt{40}}{2}$ are $\begin{pmatrix} 1+\sqrt{10} \\ 1 \end{pmatrix}$,

$$\begin{pmatrix} -(1+\sqrt{10}) \\ 1 \end{pmatrix}$$

Theorem (Diagonalization theorem): An nxn Matrix, A is diagonalizable if and only if A has linearly independent vectors.

Theorem: An nxn Matrix with distinct n eigen values is diagonalizable.

Theorem: A matrix A is diagonalizable if and only if the sum of the dimensions of the eigen spaces equal to n.This happens if and only if the characteristic polynomial factors completely in to linear factors and the dimension of the eigen space for each λ_k equals the multiplicity of λ_k.

MAIN RESULT

Verification of a square matrix diagonalizable or not by Constructing the matrices D, P

(18) Given any matrix of n th order, we find its eigen values, eigen vectors. After finding eigen vectors we construct D by placing the eigen values as the principal diagonal elements of the square matrix and remaining as zero. For each eigen value of the square matrix, we find eigen vector. We construct P by placing eigen vectors as columns of the matrix.

If $P^{-1}AP=D$ or AP=PD then the square matrix A is Diagonalizable.

Finding n^{th} Power of a Square matrix of order 2x2 when the matrix is diagonalizable.

(19) Consider the 2x2 matrix $A = \begin{pmatrix} 7 & 5 \\ 3 & 5 \end{pmatrix}$.Its Characteristic equation is $|A-\lambda I|=0$.It gives

$\begin{vmatrix} 7-\lambda & 5 \\ 3 & 5-\lambda \end{vmatrix}$ =0. By simple calucation we get eigen values as $\lambda = 10, \lambda = 2$.

We find eigen vectors by solving (A-λI)X = 0.It gives $\begin{pmatrix} 7-\lambda & 5 \\ 3 & 5-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 7-\lambda & 5 \\ 3 & 5-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\Rightarrow \begin{pmatrix} 7-10 & 5 \\ 3 & 5-10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \& \begin{pmatrix} 7-2 & 5 \\ 3 & 5-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\Rightarrow \begin{pmatrix} -3 & 5 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \& \begin{pmatrix} 5 & 5 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
⇒-3x+5y=0, 3x-5y=0and 5x+5y=0, 3x+3y=0
These equations equivalent to 3x-5y=0&x+y=0.
From the above equations we get $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \& \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

These are two eigen vectors corresponding to eigen values $\lambda = 10, \lambda = 2$.

From this we can construct the matrix P.

P=$\begin{pmatrix} 5 & 1 \\ 3 & -1 \end{pmatrix}$,$P^{-1}=\frac{-1}{8}\begin{pmatrix} -1 & -1 \\ -3 & 5 \end{pmatrix}$ or $P^{-1}=\frac{1}{8}\begin{pmatrix} 1 & 1 \\ 3 & -5 \end{pmatrix}$

As eigen values are $\lambda = 10, \lambda = 2,$ diagonal matrix $D = \begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix}$

Here AP=PD=$\begin{pmatrix} 50 & 2 \\ 30 & -2 \end{pmatrix}$ or equivalently $P^{-1}AP=D$. Hence the matrix A is Diagonalizable.

As $P^{-1}AP=D$, we can show that $P^{-1}A^kP=D^k$, k is a real number. In particular for k=$\frac{1}{2}$,

$P^{-1}A^{\frac{1}{2}}P=D^{\frac{1}{2}}$ or $P^{-1}\sqrt{A}P=\sqrt{D}$.From this we can write $\sqrt{A} = P\sqrt{D}P^{-1}$.

Hence $\sqrt{A} = \begin{pmatrix} 5 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & \frac{-5}{8} \end{pmatrix} \Rightarrow \sqrt{A} = \begin{pmatrix} \frac{5\sqrt{10}+3\sqrt{2}}{8} & \frac{5\sqrt{10}-5\sqrt{2}}{8} \\ \frac{3\sqrt{10}-3\sqrt{2}}{8} & \frac{3\sqrt{10}+5\sqrt{2}}{8} \end{pmatrix}$

Also, $A^n = \begin{pmatrix} \frac{5(10^n)+3(2^n)}{8} & \frac{5(10^n)-5(2^n)}{8} \\ \frac{3(10^n)-3(2^n)}{8} & \frac{3(10^n)+5(2^n)}{8} \end{pmatrix}$ for n is a real number.

We can also find the cube root,4th root, ----n th root of the matrix $A.\ i.e\ we\ can\ find \sqrt[3]{A}, \sqrt[4]{A}$,--------,$\sqrt[n]{A}$

By the above formula, we can find any power of the matrix A.

Verification: We can verifythe above result by multiplying $\sqrt{A} with \sqrt{A}$

$(\sqrt{A})\ (\sqrt{A}) = \begin{pmatrix} \frac{5\sqrt{10}+3\sqrt{2}}{8} & \frac{5\sqrt{10}-5\sqrt{2}}{8} \\ \frac{3\sqrt{10}-3\sqrt{2}}{8} & \frac{3\sqrt{10}+5\sqrt{2}}{8} \end{pmatrix}\begin{pmatrix} \frac{5\sqrt{10}+3\sqrt{2}}{8} & \frac{5\sqrt{10}-5\sqrt{2}}{8} \\ \frac{3\sqrt{10}-3\sqrt{2}}{8} & \frac{3\sqrt{10}+5\sqrt{2}}{8} \end{pmatrix}$

$=\frac{1}{64}$

$$\begin{pmatrix} 250+30\sqrt{20}+18+150-15\sqrt{20}-15\sqrt{20}+30 & 250-25\sqrt{20}+15\sqrt{20}-30+150+25\sqrt{20}-15\sqrt{20}-50 \\ 150+9\sqrt{20}-15\sqrt{20}-18+90-9\sqrt{20}+15\sqrt{20}-30 & 150-15\sqrt{20}-15\sqrt{20}+30+90+30\sqrt{20}+50 \end{pmatrix}$$

$=\frac{1}{64}\begin{pmatrix} 448 & 320 \\ 192 & 320 \end{pmatrix}=\begin{pmatrix} 7 & 7 \\ 3 & 5 \end{pmatrix}$=A

Hence $(\sqrt{A})\ (\sqrt{A})$=A is verified.Similarly, we can verify$(\sqrt[3]{A})^3$=A.

(20) Now we find n th power of a 3x3 Matrix.

Consider the matrix $\begin{pmatrix} \mathbf{5} & \mathbf{-8} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} \\ \mathbf{0} & \mathbf{0} & \mathbf{-2} \end{pmatrix}$

Let A=$\begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{pmatrix}$.The characteristic equation of A is $|A-\lambda I|=0$

$|A-\lambda I|=0 \Rightarrow \begin{vmatrix} 5-\lambda & -8 & 1 \\ 0 & -\lambda & 7 \\ 0 & 0 & -2-\lambda \end{vmatrix}=0$

$\Rightarrow(5-\lambda)(-\lambda)(-2-\lambda)=0 \Rightarrow 5-\lambda=0$ or $-\lambda=0$ or $-2-\lambda=0$

$\Rightarrow\lambda=5$ or $\lambda=0, \lambda=-2$ .Hence eigen values of the matrix A are 5,0, -2.

Writing the eigen values as elements of principal diagonal elements and the other elements as zero of a square matrix D, we get D=$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

We find the eigen vectors of the matrix A by solving $(A-\lambda I)X=0$.

$(A-\lambda I)X=0 \Rightarrow \begin{pmatrix} 5-\lambda & -8 & 1 \\ 0 & -\lambda & 7 \\ 0 & 0 & -2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$, where$\lambda=5$ or $\lambda=0, \lambda=-2$

$\lambda=5 \Rightarrow \begin{pmatrix} 5-5 & -8 & 1 \\ 0 & -5 & 7 \\ 0 & 0 & -2-5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0 \Rightarrow \begin{pmatrix} 0 & -8 & 1 \\ 0 & -5 & 7 \\ 0 & 0 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$

$\Rightarrow$-8y+z=0, -5y+7z=0, -7z=0 $\Rightarrow$z=0.Substituting z=0 in either -8y+z=0 or -5y+7z=0

We get y=0.We can assign any value to x. Let x=1.Now eigen vector corresponding to eigen value 5 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda=0 \Rightarrow \begin{pmatrix} 5-0 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2-0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0 \Rightarrow \begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0 \Rightarrow$5x-8y+z=0,

7z=0, -2z=0⇒5x-8y+z=0, z=0.Substituting z=0 in 5x-8y+z=0, we get 5x-8y=0.

5x-8y=0⇒5x=8y⇒x=$\frac{8}{5}$ y. By assuming y=5, we get x=8.

Hence x=8, y=5, z=0.∴ Eigen vector corresponding to the eigen value 0 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 8 \\ 5 \\ 0 \end{pmatrix}$

$\lambda=-2 \Rightarrow \begin{pmatrix} 5+2 & -8 & 1 \\ 0 & 2 & 7 \\ 0 & 0 & -2+2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0 \Rightarrow \begin{pmatrix} 7 & -8 & 1 \\ 0 & 2 & 7 \\ 0 & 0 & -0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$

⇒ 7x-8y+z=0, -2y+7z=0. 2y+7z=0⇒2y=-7z⇒y=$\frac{-7}{2}$ z. z=1 ⇒y=$-\frac{7}{2}$

Now substituting y=$\frac{-7}{2}$, z=1 in 7x-8y+z=0, we get 7x+28+1=0⇒7x=-29⇒x=$\frac{-29}{7}$

∴ x=$\frac{-29}{7}$, y=$\frac{-7}{2}$,z=1.Hence Eigen vector corresponding to the eigen value -2 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} \frac{-29}{7} \\ \frac{-7}{2} \\ 1 \end{pmatrix}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -58 \\ -49 \\ 14 \end{pmatrix}$

Hence eigen vectors corresponding to eigen values of λ =5 , λ=0,$\lambda=-2$ are

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 8 \\ 5 \\ 0 \end{pmatrix},\begin{pmatrix} -58 \\ -49 \\ 14 \end{pmatrix}$

By writing these vectors as columns of the matrix P, we get P=$\begin{pmatrix} 1 & 8 & -58 \\ 0 & 5 & -49 \\ 0 & 0 & 14 \end{pmatrix}$.

We have to verify AP=PD or P^{-1}AP=D.

The cofactors of 1,8, -58 are 70,0,0; 0,5, -49 are -112,14,0;0,0,14 are -102,49,5

Hence cofactor matrix of P=$\begin{pmatrix} 70 & 0 & 0 \\ -112 & 14 & 0 \\ -102 & 49 & 5 \end{pmatrix}$

∴ Adj P=$\begin{pmatrix} 70 & -112 & -102 \\ 0 & 14 & 49 \\ 0 & 0 & 5 \end{pmatrix}$.But det P=70

Hence $P^{-1}=\frac{1}{70}\begin{pmatrix} 70 & -112 & -102 \\ 0 & 14 & 49 \\ 0 & 0 & 5 \end{pmatrix}$ or $P^{-1}=\begin{pmatrix} 1 & \frac{-56}{35} & 0 \\ 0 & \frac{1}{5} & \frac{7}{10} \\ 0 & 0 & \frac{1}{14} \end{pmatrix}$

AP = $\begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 8 & -58 \\ 0 & 5 & -49 \\ 0 & 0 & 14 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 116 \\ 0 & 0 & 98 \\ 0 & 0 & -28 \end{pmatrix}$

PD= $\begin{pmatrix} 1 & 8 & -58 \\ 0 & 5 & -49 \\ 0 & 0 & 14 \end{pmatrix}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 116 \\ 0 & 0 & 98 \\ 0 & 0 & -28 \end{pmatrix}$

Hence AP=PD. Hence the matrix A = $\begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{pmatrix}$ is Diagonalizable.

$P^{-1}AP=D \Rightarrow P^{-1}A^KP = D^K$.From this we obtain $A^K=PD^KP^{-1}$. But D= $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

Hence $D^K = \begin{pmatrix} 5^K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2^K \end{pmatrix}$.But $A^K=PD^KP^{-1}$. But $P = \begin{pmatrix} 1 & 8 & -58 \\ 0 & 5 & -49 \\ 0 & 0 & 14 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 1 & \frac{-56}{35} & 0 \\ 0 & \frac{1}{5} & \frac{7}{10} \\ 0 & 0 & \frac{1}{14} \end{pmatrix}$

Hence $A^K = \begin{pmatrix} 1 & 8 & -58 \\ 0 & 5 & -49 \\ 0 & 0 & 14 \end{pmatrix}\begin{pmatrix} 5^K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2^K \end{pmatrix}\frac{1}{70}\begin{pmatrix} 70 & -112 & -102 \\ 0 & 14 & 49 \\ 0 & 0 & 5 \end{pmatrix}$

$\Rightarrow A^K = \begin{pmatrix} 5^K & 0 & -58(2^K) \\ 0 & 0 & 49(2^K) \\ 0 & 0 & 14(2^K) \end{pmatrix}\frac{1}{70}\begin{pmatrix} 70 & -112 & -102 \\ 0 & 14 & 49 \\ 0 & 0 & 5 \end{pmatrix}$

$\Rightarrow A^K = \frac{1}{70}\begin{pmatrix} 5^K & 0 & -58(2^K) \\ 0 & 0 & 49(2^K) \\ 0 & 0 & 14(2^K) \end{pmatrix}\begin{pmatrix} 70 & -112 & -102 \\ 0 & 14 & 49 \\ 0 & 0 & 5 \end{pmatrix}$

$\Rightarrow A^K = \frac{1}{70}\begin{pmatrix} 70(5^K) & -112(5^K) & -102(5^K) - 290(2^K) \\ 0 & 0 & 245(2^K) \\ 0 & 0 & 70(2^K) \end{pmatrix}$,where K is a real number.

From this we can find any power of A. For example,

$A^{100} = \frac{1}{70}\begin{pmatrix} 70(5^{100}) & -112(5^{100}) & -102(5^{100}) - 290(2^{100}) \\ 0 & 0 & 245(2^{100}) \\ 0 & 0 & 70(2^{100}) \end{pmatrix}$

Also, the square root of the matrix of A= $A^{\frac{1}{2}}$=

$$\sqrt{A}=\frac{1}{70}\begin{pmatrix} 70(\sqrt{5}) & -112(\sqrt{5}) & -102(\sqrt{5})-290(\sqrt{2}) \\ 0 & 0 & 245(\sqrt{2}) \\ 0 & 0 & 70(\sqrt{2}) \end{pmatrix}$$

We can find the cube root,4th root, ----n th root of the matrix $A. i.e\ we\ can\ find \sqrt[3]{A}, \sqrt[4]{A}$,--------,$\sqrt[n]{A}$.

Verification: We can verify the above result by multiplying $\sqrt{A}\ with\ \sqrt{A}$

$(\sqrt{A})(\sqrt{A})$=

$$\frac{1}{70}\begin{pmatrix} 70(\sqrt{5}) & -112(\sqrt{5}) & -102(\sqrt{5})-290(\sqrt{2}) \\ 0 & 0 & 245(\sqrt{2}) \\ 0 & 0 & 70(\sqrt{2}) \end{pmatrix}\frac{1}{70}\begin{pmatrix} 70(\sqrt{5}) & -112(\sqrt{5}) & -102(\sqrt{5})-290(\sqrt{2}) \\ 0 & 0 & 245(\sqrt{2}) \\ 0 & 0 & 70(-\sqrt{2}) \end{pmatrix}$$

$$=\frac{1}{4900}\begin{pmatrix} 24500 & -39200 & 4900 \\ 0 & 0 & 34300 \\ 0 & 0 & -9800 \end{pmatrix}=\begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{pmatrix}=\text{A}$$

Hence $(\sqrt{A})(\sqrt{A})$=A is verified.

Similarly, we can verify($\sqrt[3]{A})^3$=A.

APPLICATIONS OF DIAGONALIZATION

1. In quantum mechanics, any quantity which can be measured in a physical experiment, should be associated with Hermitian operator. For example, Hamiltonian is energy operator and it is represented by Hermitian matrix. When you diagonalize Hamiltonian in the main diagonal you will get energies of the system,
2. The stress state within an elastic solid - that which recuperates its initial shape when the forces causing its deformation stop working - can be worked out if we know the stress matrix of each point of the solid. The stress matrix diagonalization allows us to obtain the principal stresses of a specific state stress and the calculation of the equivalent stress in agreement with a criterion of fracture and / or yield.
3. We use diagonalization in solving recurrence relations and difference equations.
4. We use diagonalization in Fibonacci series
5. Vibration analysis in Engineering uses diagonalization to find natural frequencies (eigenvalues of mass and stiffness matrices), mode shapes (corresponding eigen vectors)
6. Factor analysis in statistics employs diagonalization to uncover latent variables.
7. Diagonalization is useful in machine learning and data science. It is used for spectral clustering algorithms for dimensionality reduction (projecting data on to lower dimensional spaces). It is also used for data visualization. (revealing clusters and patterns in high-dimensional data)
8. It is used for kernel methods in machine learning often involve eigen decomposition. (For support vector machines use kernel matrix diagonalization and kernel principal component analysis extends linear kernel principal component analysis to nonlinear feature spaces.)

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