Bounded Linear Operators on Hyperbolic-Valued Multi-Normed Spaces

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Abstract:
This paper introduces a componentwise operator theory for hyperbolic-valued multi-normed spaces. A quotient formula for an operator norm is generally hazardous when the denominator is in the null cone, as hyperbolic scalars contain zero divisors. The multi-bound is defined as a pair of real infimal constants, and the analysis decomposes each hyperbolic module and operator through the orthogonal idempotents. The multi-boundedness of the two component operators is demonstrated to be equivalent to multi-boundedness, which implies ordinary boundedness and continuity. Norm properties, composition estimates, kernels, ranges, inverses, and quotient-induced operators are derived. Hyperbolic statements are generated by the recombination of component versions of the open mapping, bounded inverse, closed graph, and uniform boundedness principles under Banach hypotheses. The resolvent is defined as the Cartesian product of the component resolvents. As a result, the full hyperbolic spectrum is a union of spectral cylinders rather than a coupled set of component spectral values. To differentiate between these concepts, a reduced idempotent spectrum is introduced. Diagonal, projection, and shift operators are among the examples that have been implemented. The results provide a precise indication of which classical arguments are able to withstand idempotent reduction and which necessitate additional multi-norm compatibility. Compact multi-operators, Fredholm theory, spectral radii, weak operator topologies, and hyperbolic operator inequalities are among the remaining issues.

Keywords: hyperbolic-linear operator; multi-bounded operator; operator multi-norm; closed graph theorem; uniform boundedness; hyperbolic spectrum; zero divisor

Mathematical Notation and Symbols

Symbol	Meaning
E,F	Hyperbolic normed modules $E=\mathbf{e}_1E_1\oplus\mathbf{e}_2E_2$ and $F=\mathbf{e}_1F_1\oplus\mathbf{e}_2F_2$.
$N^{(E)}_n, N^{(f)}_n$	Decomposable hyperbolic multi-norms on E^n and F^n.
$T=T_1\mathbf{e}_1+T_2\mathbf{e}_2$	Idempotent representation of a hyperbolic-linear operator.
$\lVert T\rVert mb,\mathbb{D}$	Componentwise multi-bound $\lVert T_1\rVert mb\mathbf{e}_1+\lVert T_2\rVert mb\mathbf{e}_2$.
$\mathcal{L}mb(E,F)$	Space of multi-bounded hyperbolic-linear operators.
ker T, ran T	Kernel and range of T.
$\rho\mathbb{D}(T)$, $\sigma\mathbb{D}(T)$	Full hyperbolic resolvent and spectrum.

Symbol	Meaning
σred(T)	Reduced paired idempotent spectrum.

1. Introduction

Bounded operators are the principal morphisms of normed-space analysis, while multi-bounded operators preserve the geometry encoded simultaneously at every Cartesian level. Over the hyperbolic algebra $\mathbb{D}$, the basic definition must account for two features absent over a field: nonzero zero divisors and a cone-valued norm. An expression such as N(Tx)/N(x) is not defined when N(x) has a zero component, even though x may be nonzero. The mathematically stable alternative is an inequality with a positive hyperbolic constant or, equivalently, two real operator bounds.

The product representation $\mathbb{D}\cong\mathbb{R}\times\mathbb{R}$ is decisive. A $\mathbb{D}$-linear map $T:E\to F$ commutes with the idempotent projections and therefore decomposes into real-linear maps $T_j:E_j\to F_j$. Foundational functional-analysis theorems over hyperbolic and bicomplex scalars have been established by analogous reductions (Saini et al., 2020). Multi-norm theory adds a second level of structure: ordinary boundedness concerns N_1, whereas multi-boundedness requires a constant independent of n. A bounded map need not be multi-bounded for arbitrary choices of domain and codomain multi-norms.

This paper formulates operator multi-bounds without hyperbolic division, proves the componentwise criterion, derives algebraic and Banach-space principles, and gives a precise spectral description. Established classical results are invoked only at the real component level; hyperbolic statements are presented as derived consequences of the decomposition.

2. Preliminaries

Definition 2.1. Let E and F be $\mathbb{D}$-modules. A map $T:E\to F$ is hyperbolic-linear when $T(\alpha x+\beta y)=\alpha T(x)+\beta T(y)$ for all $\alpha,\beta\in\mathbb{D}$ and $x,y\in E$.

Proposition 2.2. Every hyperbolic-linear T has a unique decomposition $T=T_1\mathbf{e}_1+T_2\mathbf{e}_2$, meaning

$$T(x_1\mathbf{e}_1+x_2\mathbf{e}_2)=T_1x_1\,\mathbf{e}_1+T_2x_2\,\mathbf{e}_2, \tag{2.1}$$

where $T_j:E_j\to F_j$ is real-linear.

Proof. Because $T(\mathbf{e}_jx)=\mathbf{e}_jT(x)$, T maps $\mathbf{e}_jE$ into $\mathbf{e}_jF$. Define T_j by the coefficient of T on that component. Additivity and real homogeneity are inherited from $\mathbb{D}$-linearity. Uniqueness follows from uniqueness of the idempotent decomposition. Conversely, Equation (2.1) defines a $\mathbb{D}$-linear map for any real-linear pair. □

Definition 2.3. Assume $N^{(E)}$ and $N^{(f)}$ are decomposable multi-norms. T is multi-bounded if there exists $C=C_1\mathbf{e}_1+C_2\mathbf{e}_2\in\mathbb{D}^+$ such that for every n and every $x_1,\dots,x_n\in E$,

$$N^{(f)}_n(Tx_1,\dots,Tx_n)\preccurlyeq_{\mathbb{D}} C\, N^{(E)}_n(x_1,\dots,x_n). \tag{2.2}$$

The admissible constants are ordered componentwise. The operator multi-norm is defined by infima, not by hyperbolic quotients:

$$\|T\|mb,\mathbb{D}=(\inf \quad C_1)\mathbf{e}_1+(\inf \tag{2.3}$$
$$C_2)\mathbf{e}_2=\|T_1\|mb\mathbf{e}_1+\|T_2\|mb\mathbf{e}_2.$$

Remark 2.4. If $N^{(E)}_n(x)$ lies in the null cone, the formal ratio $N^{(f)}_n(Tx)/N^{(E)}_n(x)$ may not exist. Equation (2.3) remains valid because each real component is optimized separately.

3. Multi-Boundedness and Continuity

Theorem 3.1. Componentwise criterion. T is multi-bounded from $(E,N^{(E)})$ to $(F,N^{(f)})$ if and only if T_j is multi-bounded from $(E_j,N^{(E)}\cdot,j)$ to $(F_j,N^{(f)}\cdot,j)$ for j=1,2. In that case Equation (2.3) holds.

Proof. Project Equation (2.2) onto $\mathbf{e}_j$. This yields $N^{(f)}_n,j(T_jx_{1j},\ldots,T_jx_{nj})\leq C_jN^{(E)}_n,j(x_{1j},\ldots,x_{nj})$ for all n, so each T_j is multi-bounded. Conversely, assume component bounds C_j. Multiply the two inequalities by $\mathbf{e}_j$ and add to obtain Equation (2.2). Taking the least admissible component constants gives Equation (2.3). □

Corollary 3.2. Every multi-bounded operator is bounded for the underlying hyperbolic norms and is globally Lipschitz continuous.

Proof. Set n=1 in Equation (2.2): $\|Tx\|\mathbb{D}\preccurlyeq\mathbb{D}\|T\|mb,\mathbb{D}\|x\|\mathbb{D}$. Then $\|T(x)-T(y)\|\mathbb{D}=\|T(x-y)\|\mathbb{D}$ has the same bound, which is continuity and uniform continuity in the component metric topology. □

Proposition 3.3. The converse in Corollary 3.2 is not automatic. It holds when the selected component multi-norms have the property that every bounded T_j is multi-bounded with $\|T_j\|mb=\|T_j\|$; in particular it holds for the minimum multi-norm $N_n(x_1,\ldots,x_n)=\max_r\|x_r\|$.

Proof. For a minimum multi-norm, $\max_r\|T_jx_r\|\leq\|T_j\|\max_r\|x_r\|$. The reverse inequality for the multi-bound follows at n=1. Component recombination proves the hyperbolic statement. For arbitrary multi-norm pairs no such estimate follows from n=1 alone, because the levels may amplify different geometric configurations. □

Proposition 3.4. If $S\in\mathcal{L}mb(F,G)$ and $T\in\mathcal{L}mb(E,F)$, then $S\circ T\in\mathcal{L}mb(E,G)$ and

$$\|S\circ T\|mb,\mathbb{D}\preccurlyeq\mathbb{D}\|S\|mb,\mathbb{D}\ \|T\|mb,\mathbb{D}. \qquad (3.1)$$

Proof. Apply the multi-bound for S to $(Tx_1,\ldots,Tx_n)$, then apply the multi-bound for T. Multiplication in $\mathbb{D}^+$ is componentwise and preserves $\preccurlyeq\mathbb{D}$. Taking component infima yields Equation (3.1). □

Proposition 3.5. The identity operator is multi-bounded with $\|I\_E\|mb,\mathbb{D}=1$, and $\mathcal{L}mb(E,E)$ is a unital algebra under addition and composition. The triangle inequality $\|S+T\|mb,\mathbb{D}\preccurlyeq\mathbb{D}\|S\|mb,\mathbb{D}+\|T\|mb,\mathbb{D}$ holds.

Proof. The identity inequality is equality at every level, so its two component bounds are one. For addition, apply the tuple triangle inequality to $((S+T)x_r)$ and then the bounds for S and T. Scalar homogeneity and closure under composition follow componentwise. □

4. Kernels, Ranges, Quotients, and Inverses

Proposition 4.1. For $T=T_1\mathbf{e}_1+T_2\mathbf{e}_2$,

$$\ker T=(\ker T_1)\mathbf{e}_1\oplus(\ker T_2)\mathbf{e}_2, \qquad \operatorname{ran} T=(\operatorname{ran} T_1)\mathbf{e}_1\oplus(\operatorname{ran} T_2)\mathbf{e}_2. \qquad (4.1)$$

Proof. Tx=0 is equivalent to $T_1x_1=0$ and $T_2x_2=0$. Every image Tx has the displayed component form, and every pair of component images is obtained by applying T to the recombined preimage. □

Corollary 4.2. T is injective, surjective, or bijective if and only if both T_1 and T_2 have the corresponding property.

Proof. Injectivity follows from the kernel identity; surjectivity from the range identity; bijectivity requires both. □

Theorem 4.3. Bounded inverse criterion. Suppose T is bijective and multi-bounded. Then T^{-1} is hyperbolic-linear. It is multi-bounded if and only if T_j^{-1} is multi-bounded for j=1,2, and then

$$\|T^{-1}\|mb,\mathbb{D}=\|T_1^{-1}\|mb\mathbf{e}_1+\|T_2^{-1}\|mb\mathbf{e}_2. \qquad (4.2)$$

Proof. Bijectivity is componentwise by Corollary 4.2, and the inverse has decomposition $T_1^{-1}\mathbf{e}_1+T_2^{-1}\mathbf{e}_2$. Apply Theorem 3.1. This statement does not infer multi-boundedness of the inverse merely from ordinary boundedness; that further step needs compatibility such as the minimum multi-norm or an appropriate inverse theorem in the component multi-operator category. □

Proposition 4.4. If T is bounded for N_1, then ker T is closed. If M⊂ker T is a closed hyperbolic submodule, the induced operator $\tilde{T}$:E/M→F, $\tilde{T}$(x+M)=Tx, is well defined and componentwise bounded under the quotient norm.

Proof. If x_m∈ker T and x_m→x, continuity gives Tx=lim Tx_m=0. Well-definedness of $\tilde{T}$ follows because representatives differ by an element of M⊂ker T. The quotient norm and boundedness reduce to the standard component quotient construction. □

5. Banach-Space Principles

In this section E and F are hyperbolic Banach modules: E_j and F_j are real Banach spaces. The topology is generated by the two component norms. The conclusions concern ordinary boundedness unless the chosen multi-norms make ordinary and multi-boundedness equivalent.

Theorem 5.1. Hyperbolic open mapping principle. A bounded surjective hyperbolic-linear operator T:E→F maps every open set to an open set.

Proof. By Corollary 4.2, T_1 and T_2 are bounded surjections. The classical open mapping theorem makes each T_j open. A basic neighborhood of x is the product of component neighborhoods; its image is the product of their open images. Hence T is open in the hyperbolic product topology. □

Corollary 5.2. If a bounded T:E→F is bijective, then T^{-1} is bounded. If domain and codomain carry minimum multi-norms, T^{-1} is also multi-bounded and $\|T^{-1}\|_{mb,\mathbb{D}}=\|T^{-1}\|_{\mathbb{D}}$.

Proof. Apply the classical bounded inverse theorem to both components, then recombine. The final assertion follows from Proposition 3.3. □

Theorem 5.3. Hyperbolic closed graph principle. A hyperbolic-linear T:E→F is bounded if and only if its graph is closed in E×F.

Proof. If T is bounded, continuity makes the graph closed. Conversely, the graph of T is closed exactly when the graphs of T_1 and T_2 are closed: convergence in E×F is componentwise. The classical closed graph theorem makes each T_j bounded, and the pair of bounds gives boundedness of T. □

Theorem 5.4. Hyperbolic uniform boundedness principle. Let {T_a:a∈A} be bounded hyperbolic-linear maps E→F. If for every x∈E the set $\{\|T_a x\|_{\mathbb{D}}:a\in A\}$ is componentwise bounded, then there is $C\in\mathbb{D}^+$ such that $\|T_a\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}} C$ for all a.

Proof. Fix $x_1\in E_1$ and embed it as $x_1\mathbf{e}_1$. Pointwise hyperbolic boundedness gives pointwise boundedness of $\{T_{a,1}\}$; the classical uniform boundedness principle yields $\sup_a\|T_{a,1}\|<\infty$. Repeat for the second component. Combining the two suprema gives C. □

Corollary 5.5. If all component spaces carry minimum multi-norms, pointwise boundedness in Theorem 5.4 implies a uniform bound on $\|T_a\|_{mb,\mathbb{D}}$.

Proof. For minimum multi-norms, component operator norm and multi-bound coincide by Proposition 3.3. □

6. Hyperbolic Resolvent and Spectrum

Definition 6.1. For $T\in\mathcal{L}(E)$, the full hyperbolic resolvent is $\rho_{\mathbb{D}}(T)=\{\lambda\in\mathbb{D}:\lambda I-T$ is bijective with bounded inverse$\}$; the full spectrum is $\sigma_{\mathbb{D}}(T)=\mathbb{D}\setminus\rho_{\mathbb{D}}(T)$.

Theorem 6.2. Resolvent product formula. For $\lambda=\lambda_1\mathbf{e}_1+\lambda_2\mathbf{e}_2$,

$$\lambda\in\rho_{\mathbb{D}}(T) \Leftrightarrow \lambda_1\in\rho(T_1) \text{ and } \lambda_2\in\rho(T_2), \quad (6.1)$$

$$R(\lambda,T)=R(\lambda_1,T_1)\mathbf{e}_1+R(\lambda_2,T_2)\mathbf{e}_2. \quad (6.2)$$

Proof. The operator $\lambda I-T$ decomposes as $(\lambda_1 I-T_1)\mathbf{e}_1+(\lambda_2 I-T_2)\mathbf{e}_2$. It is bijective with bounded inverse precisely when both components are. Multiplication of the proposed inverse by $\lambda I-T$ gives I componentwise. □

Corollary 6.3. The full spectrum is the union of spectral cylinders

$$\sigma_{\mathbb{D}}(T)=[\sigma(T_1)\mathbf{e}_1+\mathbb{R}\mathbf{e}_2] \cup [\mathbb{R}\mathbf{e}_1+\sigma(T_2)\mathbf{e}_2]. \quad (6.3)$$

Proof. Take the complement in $\mathbb{R}\times\mathbb{R}$ of $\rho(T_1)\times\rho(T_2)$. The complement of a Cartesian product is $(\sigma(T_1)\times\mathbb{R})\cup(\mathbb{R}\times\sigma(T_2))$, which is Equation (6.3). □

Definition 6.4. The reduced idempotent spectrum is $\sigma_{red}(T)=\sigma(T_1)\mathbf{e}_1+\sigma(T_2)\mathbf{e}_2$. It records paired component spectral values but is generally a proper subset of the full spectrum.

Remark 6.5. The distinction is forced by zero divisors. If $\lambda_1\in\sigma(T_1)$ but $\lambda_2\in\rho(T_2)$, then $\lambda I-T$ fails to be invertible because its first component fails, even though its second component has a resolvent. Such λ lies in a full spectral cylinder but not necessarily in $\sigma_{red}(T)$.

Proposition 6.6. The resolvent identity holds on $\rho_{\mathbb{D}}(T)$: for $\lambda,\mu\in\rho_{\mathbb{D}}(T)$,

$$R(\lambda,T)-R(\mu,T)=(\mu-\lambda)R(\lambda,T)R(\mu,T). \quad (6.4)$$

Proof. Apply the classical resolvent identity in each real component and recombine. The formula multiplies by $\mu-\lambda$ but does not divide by it, so it remains meaningful even when $\mu-\lambda$ is a zero divisor. □

7. Worked Operator Examples

Example 7.1. Diagonal operator on $\mathbb{D}^m$. Let $A=A_1\mathbf{e}_1+A_2\mathbf{e}_2$ with real matrices A_j and $T(x)=Ax$. Under component Euclidean minimum multi-norms, $\|T\|_{mb,\mathbb{D}}=\|A_1\|_2\mathbf{e}_1+\|A_2\|_2\mathbf{e}_2$. T is invertible exactly when det A_1 and det A_2 are nonzero.

Proof. The minimum multi-bound equals the usual induced norm in each component. Matrix multiplication and invertibility are componentwise. If one determinant vanishes, A is a zero divisor in the matrix algebra and no two-sided inverse exists. □

Example 7.2. Idempotent projection $P(x)=\mathbf{e}_1x$ has $P_1=I$ and $P_2=0$. Hence $\|P\|_{mb,\mathbb{D}}=1\cdot\mathbf{e}_1+0\cdot\mathbf{e}_2$, a nonzero zero divisor in $\mathbb{D}^+$. Its kernel is $\mathbf{e}_2E_2$ and its range is $\mathbf{e}_1E_1$.

Proof. All claims follow from direct component calculation. The example shows that a legitimate operator norm can lie in the null cone; treating nonzero norm values as invertible would therefore be incorrect. □

Example 7.3. On $\ell^p(\mathbb{D})=\mathbf{e}_1\ell^p(\mathbb{R})\oplus\mathbf{e}_2\ell^p(\mathbb{R})$, define the right shift $S(x_1,x_2,\ldots)= (0,x_1,x_2,\ldots)$. With minimum tuple multi-norms, $\|S\|_{mb,\mathbb{D}}=1$ and S is an isometry but not surjective.

Proof. Each component right shift preserves the ℓ^p norm and has range consisting of sequences with zero first coordinate. The component multi-bounds are one. □

7.4 Neumann Series, Lower Bounds, and Spectral Radius

Theorem 7.4. Hyperbolic Neumann theorem. Let E be a hyperbolic Banach module and $T\in\mathcal{L}(E)$. If $\|T\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}c=c_1\mathbf{e}_1+c_2\mathbf{e}_2$ with $0\le c_1,c_2<1$, then $I-T$ is invertible and

$$(I-T)^{-1}=\Sigma_{m=0}^{\infty}T^m, \quad (7.1)$$

$$\|(I-T)^{-1}\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}(1-c_1)^{-1}\mathbf{e}_1+(1-c_2)^{-1}\mathbf{e}_2.$$

Proof. Each component norm satisfies $\|T_j\|\le c_j<1$, so the complex or real Neumann series ΣT_j^m converges in operator norm and is the inverse of $I-T_j$. The paired series converges in the product operator norm. Geometric-series bounds give the stated coefficients. Both $1-c_j$ are positive, so the hyperbolic bound is invertible; no null-cone division occurs. □

Example 7.5. One-channel contraction is insufficient. Let $T=0\cdot\mathbf{e}_1+2I\cdot\mathbf{e}_2$. Its norm is $2\mathbf{e}_2$, which is small in the first component but not in the second. The Neumann series for $I-T$ has second component $\Sigma 2^m I$ and diverges, although $I-T$ itself happens to be invertible with second component $-I$.

Proof. The example separates a sufficient series criterion from invertibility. The second geometric series diverges, so Theorem 7.4 cannot apply. Directly, $I-T=I\mathbf{e}_1-I\mathbf{e}_2$ has inverse itself. Thus failure of the norm condition does not imply noninvertibility, but satisfying it in only one channel cannot justify the series. □

Proposition 7.6. Bounded-below criterion. T is bounded below by $m=m_1\mathbf{e}_1+m_2\mathbf{e}_2$ with $m_1,m_2>0$ when $\|Tx\|_{\mathbb{D}}\succcurlyeq_{\mathbb{D}} m\|x\|_{\mathbb{D}}$ for all x. This holds if and only if both T_j are bounded below. Then T is injective and ran T is closed.

Proof. The inequality is componentwise. A positive lower coefficient makes each T_j injective. If Tx_n converges, the lower bound makes x_n Cauchy; completeness gives x_n→x and continuity gives Tx equal to the limit. Hence the range is closed. □

Corollary 7.7. A bijective bounded-below T has bounded inverse on its range with $\|T^{-1}\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}} m_1^{-1}\mathbf{e}_1+m_2^{-1}\mathbf{e}_2$.

Proof. For y=Tx, the lower inequality gives $\|x\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}} m^{-1}\|y\|_{\mathbb{D}}$ componentwise. □

Definition 7.8. The reduced spectral radius is $r\_red(T)=r(T_1)\mathbf{e}_1+r(T_2)\mathbf{e}_2$. The full cylinder spectrum has no finite scalar radius in the usual sense because each cylinder is unbounded in the unused coordinate.

Theorem 7.9. Component spectral-radius formula. For a bounded T,

$$r\_red(T)=\lim_{n\to\infty}\|T^n\|_{\mathbb{D}}^{1/n}=\inf_{n\geq 1}\|T^n\|_{\mathbb{D}}^{1/n}, \qquad (7.2)$$

Proof. Powers and norms decompose as $T^n=T_1^n\mathbf{e}_1+T_2^n\mathbf{e}_2$ and $\|T^n\|_{\mathbb{D}}=\|T_1^n\|\mathbf{e}_1+\|T_2^n\|\mathbf{e}_2$. Apply the classical spectral-radius formula in both components and take component roots. The result concerns the reduced radius, not a bounded radius of the full spectrum. □

Proposition 7.10. If λ belongs to the full resolvent and μ is sufficiently close componentwise so that $\|(\mu-\lambda)R(\lambda,T)\|_{\mathbb{D}}\prec_{\mathbb{D}}1$, then μ belongs to the resolvent and

$$R(\mu,T)=R(\lambda,T)\Sigma_{n=0}^{\infty}[(\lambda-\mu)R(\lambda,T)]^n. \qquad (7.3)$$

Proof. Factor $\mu I-T=(\lambda I-T)[I+(\mu-\lambda)R(\lambda,T)]$. The bracket is invertible by Theorem 7.4. Multiplying the inverse factors gives the series. This proves openness of the resolvent in the product topology and component analyticity of the resolvent map. □

These results show why operator inequalities must use two strict conditions. A scalarization such as $\max(\|T_1\|,\|T_2\|)$ can provide a sufficient real criterion, but it suppresses asymmetric estimates. The hyperbolic form records separate convergence radii, lower bounds, and resolvent neighborhoods. For multi-bounded operators under minimum multi-norms, the same formulas hold with the multi-bound because ordinary and multi-operator norms coincide.

7.11 Block Operators and Direct Sums

Proposition 7.11. Let $T\in\mathcal{L}mb(E)$ and $S\in\mathcal{L}mb(F)$, and give $E\oplus F$ the product multi-norm $P_n=\max_{\mathbb{D}}\{N_n^E,N_n^F\}$. The block-diagonal operator $A=T\oplus S$ is multi-bounded with

$$\|A\|_{mb,\mathbb{D}}=\max_{\mathbb{D}}\{\|T\|_{mb,\mathbb{D}},\|S\|_{mb,\mathbb{D}}\}. \qquad (7.4)$$

Proof. At every level, $P_n(A(x_r,y_r))$ is the maximum of $N_n^E(Tx_r)$ and $N_n^F(Sy_r)$, bounded by the maximum of the two operator constants times $P_n((x_r,y_r))$. Equality follows by testing pure E and pure F tuples. □

Proposition 7.12. For bounded block diagonal $A=T\oplus S$, the component spectrum satisfies $\sigma(A_j)=\sigma(T_j)\cup\sigma(S_j)$. Therefore the full hyperbolic spectrum is obtained by forming cylinders over these component unions.

Proof. The operator $\lambda_j I-A_j$ is the direct sum of $\lambda_j I-T_j$ and $\lambda_j I-S_j$ and is invertible exactly when both summands are invertible. The classical direct-sum spectral identity follows, and the hyperbolic cylinder formula then applies. □

Block operators provide a convenient language for coupled equations even when the coupling is initially absent. Off-diagonal perturbations can then be studied by a Neumann argument: if $A_0=T\oplus S$ is invertible and K is an off-diagonal bounded operator with $\|A_0^{-1}K\|\mathbb{D}\prec\mathbb{D}1$, then $A_0+K=A_0(I+A_0^{-1}K)$ is invertible. The two component inequalities quantify how much coupling each channel can tolerate. This is preferable to a single averaged condition when one idempotent component is substantially more ill-conditioned than the other.

8. Applications and Interpretive Analysis

Two simultaneous estimates are encoded by hyperbolic operator bounds. In a split-channel evolution, where $x_{m+1}=Tx_m$, the inequality $\|T^n\|\mathbb{D}\preccurlyeq\mathbb{D}\|T\|\mathbb{D}^n$ results in distinct growth rates. The product order maintains this information, as one component may decompose while the other expands. A Neumann-series inverse can be constructed componentwise for iterative equations $x=Tx+b$ when both component norms are below one. A single scalarized average below one is insufficient.

The action of T on arbitrary finite families is regulated by multi-boundedness, which is characterised by a single constant that is independent of the size of the family. This is more robust than pointwise continuity, unless the multi-norms are operator stable, as is the case with the minimum structure. Operator-generated sequence spaces, vector-valued integration, and the stability of functional equations may all benefit from the application of such bounds. Hyperbolic spectral cylinders are pertinent when the operative concept is invertibility, rather than a designated paired eigenvalue notion.

The two real operator problems should be expressly stored in the implementation of computational mathematics. The resolution of a hyperbolic linear system necessitates the resolution of both component systems; the invertibility of the entire system is rendered impossible by the failure of either component. This rule prevents numerical attempts to divide by null-cone quantities and ensures that condition estimates are transparent.

9. Open Problems and Limitations

Ordinary boundedness is the focus of the Banach-space principles mentioned above. Their direct multi-bounded analogues necessitate the completeness of an operator space under the multi-bound and compatibility assumptions, which are not automatic. A systematic theory of compact, weakly compact, nuclear, and absolutely summing hyperbolic multi-operators is yet to be developed from the corresponding component ideals.

Spectral enquiries encompass a practical spectral radius concept for the complete cylinder spectrum, Fredholm alternatives, essential spectra, and functional calculi that explicitly indicate whether the full or reduced spectrum is employed. Other issues include hyperbolic C*-type structures, closed unbounded operators, semigroups, and operator inequalities in the partial order. Finally, coupled hyperbolic multi-norms that are not part of the idempotent-additive class may not be reduced to two standard operator problems and would necessitate the development of authentically new methods.

10. In summary

Decomposable hyperbolic multi-norms transform multi-boundedness into a pair of classical multi-boundedness conditions, and hyperbolic-linear operators decompose uniquely into two real-linear operators. The unlawful division by zero divisors is prevented by defining the operator multi-norm through component infima. Multi-bounded operators are continuous, have componentwise kernels, ranges, and inverses, and compose submultiplicatively. Open mapping, uniform boundedness, bounded inverse, and closed graph principles are derived from the two real theorems under Banach hypotheses. The full spectrum is a union of cylinders, as the resolvent is a Cartesian product. The smaller paired spectrum serves a distinct descriptive purpose. These findings provide a precise foundation for sequence-space applications and continuous bicomplex operators.

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