Common Fixed- Point Theorem in M Fuzzy Metric Spaces with Property (E) Satisfying Integral Type Inequality

Neetu Dave[1], Dr. Arvind Gupta[2], Dr. Anupama Gupta[3], Dr. Praveen Shrivastava[4]

[1]Research Scholar, [2,3,4]Professor
Department of Mathematics
Govt. MVM college Bhopal (M.P) India.

Abstract:
The aim of this paper is to established common fixed-point theorem in M fuzzy metric spaces for property (E) using integral type inequality.

Keywords: weakly compatible maps, fixed-point, M-fuzzy metric space.

1.INTRODUCTION:

The fuzzy set notion was first introduced by Zadeh [20] in 1965. Kramosil and Michalek [8] then created the fuzzy metric space concept, which George and Veermani [4] further refined. Numerous researchers have applied distinct mathematical results in diverse ways to fuzzy metric space [1, 2, 3]. Sessa [17] introduced the idea of a weak commuting property, which enhanced the commutative condition in the fixed-point theorem. Jungck first proposed the idea of compatible mappings for self-maps in 1986.
Sedghi and Shobe [16] proposed m-fuzzy metricspace in 2006 and established the common fixed- point theorem for weakly compatible mappings, which is a generalization of fuzzy metric spaces.

2.PRELIMINARIES:

Definition 2.1: A binary operation *: [0, 1] × [0, 1] → [0, 1] is a continuous t – norm if satisfying condition:
(a) * is commutative and associative;
(b) Is continuous;
(c) a * 1 = a for all a ϵ [0, 1];
(d) a * b ≤ c * d whenever a ≤ c and b ≤ d and a, b, c and d ϵ [0, 1]

Definition 2.2: A 3-tuple $(X, M,*)$ is called a M- fuzzy metric space if X is an arbitrary (non-empty) set, * is a continuous t- norm, and M is fuzzy sets on $X^3 \times (0, \infty)$, satisfying the following conditions: for each $x, y, z, a \in X$ and $t, s > 0$
(1) $M(x, y, z, t) > 0$;
(2) $M(x, y, z, t) = 1$ if and only if $x = y = z$;
(3) $M(x, y, z, t) = M(p\{x, y, z\}, t)$, (symmetry) where p is a permutation function;
(4) $M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$;

Remark 2.3: Let$(X, M,*)$ be a M-fuzzy metric space. Then for every $t > 0$ and for every $x, y \in X$, we have $M(x, x, y, t) = M(x, y, y, t)$ Because for each $\varepsilon > 0$ by triangular inequality we have
(i) $M(x, x, y, \varepsilon + t) \geq M(x, x, x, \varepsilon) * M(x, y, y, t) = M(x, y, y, t)$
(ii) $M(y, y, x, \varepsilon + t) \geq M(y, y, y, \varepsilon) * M(y, x, x, t) = M(y, x, x, t)$.
By taking limits of (i) and (ii) when $\varepsilon \to 0$, we obtain $M(x, x, y, t) = M(x, y, y, t)$.

Definition 2.4: Let $(X, M,*)$ be a M −fuzzy metric space. For $t > 0$, the
(1) open ball $BM(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by $BM(x, r, t) = \{y \in X : M(x, y, y, t) > 1 - r\}$.
(2) A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $BM(x, r, t) \subseteq A$.

(3) A sequence $\{ x_n \}$ in X converges to x if and only if $M(x, x, x_n, t) \to 1\ as\ n \to \infty, for\ each\ t > 0$

Definition 2.5: Let A and S be mappings from a M-fuzzy metric space $(X, M,*)$ into itself Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 2.6: Let A and S be mappings from a M-fuzzy metric space $(X, M,*)$ into itself. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} M(ASx_n, SAx_n, SAx_n, t) = 1, \forall\, t > 0$$

whenever $\{ x_n \}$ is a sequence in $X\ and\ x \in X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$

Lemma 2.7: Let $(X, M,*)$ be a M −fuzzy metric space. Then $M(x, y, z, t)$ is non-decreasing with respect to $t, for\ all\ x, y, z\ in\ X$

Lemma 2.8: Let $(X, M,*)$ be a M-fuzzy metric space and for all $x, y \in X, t > 0$ and if for a number $k \in (0,1)\ such\ that\ M(x, y, kt) \geq M(x, y, t)\ then\ x = y$.

Definition 2.9: Let (X, d) be a compatible metric space, $c \in [0,1)$, $f : X \to X$ a mapping such that for each $x, y \in X$

$$\int_0^{d\,(fx,\ fy)} \psi(\mathrm{t})\mathrm{dt} \leq \int_0^{d\,(x,\ y)} \psi(\mathrm{t})\mathrm{dt}$$

where $\psi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, nonnegative, and such that, for each $\varepsilon > 0, \int_0^{\varepsilon} \psi(\mathrm{t})\mathrm{dt} > 0$,then f has a unique common fixed z ∈ X such that for each $x \in X, \log_{n\to\infty} f^n x = z$.

Definition 2.10: An element $x \in X$ is called a coincidence point of the mapping $f: X \to X\ and\ g : X \to X\ if\ f\,(x) = g(x)\ and\ f\,(y) = g(y)$

Definition 2.11: Let A and B be a mappings from a M- fuzzy metric space $(X, M,*)$ into itself, then the mapping are said to be weak compatible if they commute at their coincidence point that is $Ax = Sx$, implies that $ASx = SAx$.

Definition 2.12: Let A and B be a mapping from a M- fuzzy metric space $(X, M,*)$ into itself, then the mapping is said to be compatible if

$\log_{n\to\infty} M(ASx_n, SAx_n, SAx_n, t) = 1, \ \forall\, \mathrm{t} > 0$

whenever $\{ x_n \}$ is a sequence in $X\ and\ x \in X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$.

Example 2.13: Let $X = R$ (set of real number) define two functions: $f(x) = 2x, g(x) = x$

We want to find points $x \in R where f(x) = g(x)$;

$2x = x \Rightarrow x = 0$, so, the only coincidence is = 0

Now, $f(g(x)) = g(f(x))\ at\ x = 0$, so $f\ and\ g$ commute at their coincidence point, functions $f\ and\ g$ are weakly compatible because they commute at their only coincidence point x=0.

Example 2.14: Let $X = R$ and $M(x, y, z, t) = \frac{t}{t + |x-y| + |y-z| + |z-x|}$

For every x, y, z ∈ X and t > 0. Clearly that $(X, M\,,*)$ is M- fuzzy metric space, Let A and B

Defined by $Ax = \frac{\sqrt{1-(2x-1)^2}}{2}$ and $Bx = (1 - x)$

Here A and B has two coincidence points $x = 1, x = 1/2$, since $A1 = B1 = 0$ for $x = 1\ also\ for\ x = 1/2$ we have $A½ = B½ = ½$ and a common fixed- point $x = 1/2$. So, $A\ and\ B$ are owc maps, since they commute at one of their coincidence points $x = ½$.

Definition 2.15: Let A and B be two self- mappings of a M fuzzy metric space $(X, M,*)$, we say that A and B satisfy the property (E), if there exists a sequence $\{x_n\}$ such that

$\lim_{n\to\infty} M(Ax_n, u, u, t) = \lim_{n\to\infty} M(Bx_n, u, u, t) = 1$, for some $u \in X\ and\ t > 0$.

Example 2.16: Let $X = R$ and $M(x, y, z, t) = \frac{t}{t + |x-y| + |y-z| + |z-x|}$

For every x, y, z ∈ X and t > 0, Let A and B defined $Ax = 2x + 1, Bx = x + 2$.

Consider the sequence $x_n = \frac{1}{n} + 1, n = 1,2,3 \ldots\ldots\ldots. thus\ we\ have$

$\lim_{n\to\infty} M(Ax_n, 3\ , 3, t) = \lim_{n\to\infty} M(Bx_n,\ 3, 3, t) = 1$, for every $t > 0$. Then $A\ and\ B$ satisfying in the property (E).

3.MAIN RESULT:

Theorem 3.1: Let A, B, S and T be a self-mapping on a fuzzy metric space $(X, M,*)$,satisfying the following conditions:

(i) The pair (A, S) and (B, T) are weakly compatible and (A, S) or (B, T) satisfy the property (E),

(ii) $A(X) \subseteq T(X), B(X) \subseteq S(X)$ and $T(X)$ or $S(X)$ is a complete fuzzy metric subspace of X. If there exist $0 < q < 1/2$ and $t > 0$ $such\ that$

$$\int_0^{M(Ax,By,Bz,qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(Sx,Ty,Tz,t), M(Ty,By,Bz,t), M(Ty,Ty,Tz,t), \\ M(Sx,By,Bz,\alpha t), M(Tz,Bz,Bz,(2-\alpha)t \end{pmatrix}} \psi(t)dt \quad (3.1a)$$

For all $0 < \alpha < 2$ and $x, y \in X$ then A, B, S, T have a unique common fixed-point in X.

Proof: Let (B, T) satisfy the property (E) , then there exist a sequence $\{x_n\}$ such that
$\lim_{n\to\infty} M(Bx_n, u, u, t) = \lim_{n\to\infty} M(Tx_n, u, u, t) = 1$
For some $u \in X$ and $t > 0$. Since $BX \subseteq SX$, then there exists a sequence y_n such that, $Bx_n = Sy_n$ hence $\lim_{n\to\infty} M(Bx_n, u, u, t) = 1$.
We prove that $\lim_{n\to\infty} M(Ay_n, u, u, t) = 1$.
With the help of inequality (3.1a) we have

$$\int_0^{M(Ay_n,Bx_n,Bx_{n+1},qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(Sy_n,Tx_n,Tx_{n+1},t),\ M(Tx_n,Bx_n,Bx_{n+1},t),\ M(Tx_n,Tx_n,Tx_{n+1},t), \\ M(Sy_n,Bx_n,Bx_{n+1},\alpha t),\ M(Tx_{n+1},Bx_{n+1},Bx_{n+1},(2-\alpha)t \end{pmatrix}} \psi(t)dt$$

On taking limit $n \to \infty$ we have,

$$\int_0^{M(Ay_n,u,u,qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(u,u,u,t),\ M(u,u,u,t),\ M(u,u,u,t), \\ M(u,u,u,\alpha t),\ M(u,u,u,(2-\alpha)t \end{pmatrix}} \psi(t)dt$$

taking $\alpha = 1 - \beta$ where $\beta \in (0,1)$ then we have,

$$= \int_0^{min\begin{pmatrix} M(u,u,u,t),\ M(u,u,u,t),\ M(u,u,u,t), \\ M(u,u,u,(1-\beta)t),\ M(u,u,u,(1+\beta)t \end{pmatrix}} \psi(t)dt$$

On taking limit $\beta \to \infty$,

$$= \int_0^{min\begin{pmatrix} M(u,u,u,t),\ M(u,u,u,t),\ M(u,u,u,t), \\ M(u,u,u,t),\ M(u,u,u,t) \end{pmatrix}} \psi(t)dt$$

$$\int_0^{M(Ay_n,u,u,qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(u,u,u,t),\ M(u,u,u,t),\ M(u,u,u,t), \\ M(u,u,u,t),\ M(u,u,u,t \end{pmatrix}} \psi(t)dt$$

Since $min\{M(u, u, u, t), M(u, u, u, t), M(u, u, u, t), M(u, u, u, t), M(u, u, u, t)\} = M(u, u, u, t)$ therefore,

$$\int_0^{M(Ay_n,u,u,qt)} \psi(t)dt \geq \int_0^{(M(u,u,u,t)} \psi(t)dt$$

Since $M(u, u, u, t) = 1$
Therefore $\lim_{n\to\infty} M(Ay_n, u, u, t) = 1$.
Hence $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = u$.
Let $S(X)$ be complete M −fuzzy metric space, then there exists an element $x \in X$ such that $Sx = u$. if $Ax \neq u$,then we have, by inequality (3.1a)

$$\int_0^{M(Ax,Bx_n,Bx_{n+1},qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(Sx,Tx_n,Tx_{n+1},t), M(Tx_n,Bx_n,Bx_{n+1},t), M(Tx_n,Tx_n,Tx_{n+1},t), \\ M(Sx,Bx_n,Bx_{n+1},\alpha t), M(Tx_{n+1},Bx_{n+1},Bx_{n+1},(2-\alpha)t \end{pmatrix}} \psi(t)dt$$

On taking limit $n \to \infty$, and put $\alpha = 1 - \beta$ $where$ $\beta \in (0,1)$

$$\int_0^{M(Ax,u,u,qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(u,u,u,t),\ M(u,u,u,t),\ M(u,u,u,t), \\ M(u,u,u,(1-\beta)t),\ M(u,u,u,(1+\beta)t \end{pmatrix}} \psi(t)dt$$

now taking limit $\beta \to \infty$, we obtain

$$\int_0^{M(Ax,u,u,qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(u,u,u,t),\ M(u,u,u,t),\ M(u,u,u,t), \\ M(u,u,u,t),\ M(u,u,u,t) \end{pmatrix}} \psi(t)dt$$

Since $min\{M(u, u, u, t), M(u, u, u, t), M(u, u, u, t), M(u, u, u, t), M(u, u, u, t)\} = M(u, u, u, t)$ therefore,

$$\int_0^{M(Ax,u,u,qt)} \psi(t)dt \geq \int_0^{(M(u,u,u,t)} \psi(t)dt$$

Since $M(u, u, u, t) = 1$

Therefore $\lim_{n\to\infty} M(Ax, u,u,t) = 1.$, hence $Ax = u = Sx$, since (A,S) be a weakly compatible which implies$ASx = SAx$, that means $AAx = ASx = SAx = SSx$.
As $AX \subseteq TX$, then there exists $w \in X$ such that $Ax = Tw$, now e proves that $Tw = Bw$. If $Tw \neq Bw$ then by inequality (3.1a)

$$\int_0^{M(Ax,Bw,Bw,qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(Sx,Tw,Tw,t),M(Tw,Bw,Bw,t),M(Tw,Tw,Tw,t), \\ M(Sx,Bw,Bw,\alpha t),M(Tw,Bw,Bw,(2-\alpha)t \end{pmatrix}} \psi(t)dt$$

Put $\alpha = 1-\beta\ where\ \beta \in (0,1)$ and taking limit $\beta \to \infty$

$$\int_0^{M(Ax,Bw,Bw,qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(Sx,Tw,Tw,t),M(Tw,Bw,Bw,t),M(Tw,Tw,Tw,t), \\ M(Sx,Bw,Bw,t),M(Tw,Bw,Bw,t) \end{pmatrix}} \psi(t)dt$$

If $Bw \neq u$ then we have,

$$\int_0^{M(Ax,Bw,Bw,qt)} \psi(t)dt \geq \int_0^{min(M(Ax,Bw,Bw,t))} \psi(t)dt$$

Here contradiction implies that $Tw = Bw = u$, As the pair (B,T) is weakly compatible, we get

$$TTw = TBw = BTw = BBw, so\ Tu = Bu$$

Now we prove $Au = u$, for

$$\int_0^{M(Au,u,u,qt)} \psi(t)dt = \int_0^{M(Au,Bw,Bw,qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(Su,Tw,Tw,t),M(Tw,Bw,Bw,t),M(Tw,Tw,Tw,t), \\ M(Su,Bw,Bw,\alpha t),M(Tw,Bw,Bw,(2-\alpha)t \end{pmatrix}} \psi(t)dt$$

$$\Rightarrow \int_0^{M(Au,u,u,qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(Su,u,u,t),M(u,u,u,t),M(u,u,u,t), \\ M(Su,u,u,\alpha t),M(u,u,u,(2-\alpha)t \end{pmatrix}} \psi(t)dt$$

on taking $\alpha = 1-\beta\ where\ \beta \in (0,1)$ and taking limit $\beta \to \infty$,

$$\int_0^{M(Au,u,u,qt)} \psi(t)dt \geq \int_0^{min(M(Au,u,u,t)} \psi(t)dt$$

Is a contradiction thus $Au = Su = u$.
Similarly ,we can prove $Tu = Bu = u$. or we can write $Au = Su = Tu = Bu = u$. Thus A,B,S,T have a common fixed -point u.
Uniqueness: Let v be another common fixed-point of $A,B\ S\ and\ T$. then by inequality (3.1a)

$$\int_0^{M(v,u,u,qt)} \psi(t)dt = \int_0^{M(Au,Bu,Bu,qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(Sv,Tu,Tu,t),M(Tu,Bu,Bu,t),M(Tu,Tu,Tu,t), \\ M(Sv,Bu,Bu,\alpha t),M(Tu,Bu,Bu,(2-\alpha)t) \end{pmatrix}} \psi(t)dt$$

on taking $\alpha = 1-\beta\ where\ \beta \in (0,1)$ and taking limit $\beta \to \infty$,

$$\int_0^{M(v,u,u,qt)} \psi(t)dt \geq \int_0^{min\begin{pmatrix} M(Sv,Tu,Tu,t),M(Tu,Bu,Bu,t),M(Tu,Tu,Tu,t), \\ M(Sv,Bu,Bu,t),M(Tu,Bu,Bu,t) \end{pmatrix}} \psi(t)dt$$

$$\Rightarrow \quad \int_0^{M(v,u,u,qt)} \psi(t)dt \geq \int_0^{min(M(v,u,u,t)} \psi(t)dt$$

Is a contradiction, thus $u = v$.

REFERENCES:

[1] C. Alaca, D. Turkoglu, C. Yildiz, Fixed point in fuzzy metric spaces, Chaos Solitons Fractals 29 (2006), 1073-1078.
[2] K. Atanassov, fuzzy sets, Fuzzy Sets and Systems 20 (1986), 87-96.
[3] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type. Int. J. Math. Sci. 29(2002), 531 - 536.
[4] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst. 64 (1994), 395-399.
[5] X. Jin, Y. Piao, Common fixed points for Two Contractive Mappings of Integral Type in Metric spaces, Appl. Math. 6 (2015), 1009-1016.
[6] G. Jungck, B. E. Rhoades, Fixed point for set valued functions without continuity, Ind. J. Pure Appl. Math. 29(1998), 227-238.
[7] G. Jungck and B.E. Rhoades, Fixed point theorems for occasionally weakly compatible mappings, (Erratum), Fixed Point Theory, 9 (2008), 383-384.
[8] I. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika, 11 (1975), 336-344.

[9] S. Kutukcu, A common fixed- point theorem for a sequence of self- maps in fuzzy metric spaces, Commun. KoreanMath. Soc. 21 (2006), 679-687.
[10] Z.Q. Liu, X. Li, S.M. Kang, S.Y. Cho, Fixed point Theorems for mapping satisfying contractive conditions of Integral Type and Applications, Fixed Point Theory Appl. 2011 (2011), 64.
[11] S.V.R. Naidu, K.P.R. Rao, N. Srinivasa Rao, On the topology of D-metric spaces and the generation of D-metric spaces from metric spaces, Int. J. Math. Math. Sci. 51(2004), 2719-2740.
[12] J. H. Park, fuzzy metric spaces, Chaos Solitons Fractals. 22 (2004), 1039-1046.
[13] J.H. Park, J.S. Park, Y.C. Kwun, Fixed points in M-fuzzy metric spaces, Fuzzy Optim. Decis. Making, 7 (2008), 305-315.
[14] K.P.R. Rao, G. Ravi Babu, V.C.C. Raju, A Common Fixed Point Theorem for Three Pairs of Maps in M-Fuzzy Metric Spaces, Int. J. Contemp. Math. Sci. 3 (2008), 713 –720.
[15] B.E. Rhoades, Two fixed point theorems for mappings satisfying a general contractive condition of integral type. Int. J. Math. Math. Sci. 63 (2003), 4007 - 4013.
[16] S. Sedghi, Shobe, Fixed point theorem in M-fuzzy metric spaces with property (E), Adv. Fuzzy Math. 1 (2006), 55-65.
[17] S. Sessa, On Weak Commutativity Condition of Mapping in Fixed Point Consideration, Publ. Inst. Math (Beograd) N.S. 32 (1982), 149-153,
[18] S. Sharma, On fuzzy metric space, Southeast Asian Bull. Math. 6 (2002), 145-157.
[19] B. Schweizer, A. Sklar, Statistical metric spaces, Pac. J. Math. 10 (1960), 313-334.
[20] L.A. Zadeh, Fuzzy sets, Inform. Control. 8 (1965), 338-353.