Homomorphisms on Bicomplex Modules and Their Algebraic Properties

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Abstract:
Modules over a commutative ring with two orthogonal central idempotents and numerous zero divisors are known as bicomplex modules. Consequently, their homomorphism theory is more comprehensive in terms of terminology, but it is structurally reducible to paired complex-linear algebra. This paper provides a self-contained algebraic treatment of bicomplex-module homomorphisms. A unique pair of complex-linear mappings is demonstrated for each homomorphism $\varphi:M\to N$, and each module M is decomposed as $M=e_1M_1\oplus e_2M_2$. Kernels, images, submodules, quotients, direct sums, exact sequences, endomorphisms, and automorphisms are characterised componentwise. Proofs for the correspondence theorem and the first, second, and third isomorphism theorems are provided in detail. Despite the fact that a module does not need to be free, every brief exact sequence divides because the bicomplex ring is isomorphic to a product of two fields. The minimum number of generators is the larger component dimension, and a finite module is free of rank r precisely when both component dimensions equal r. This results in a rank vector that is more informative than a singular rank. The simplistic concept of torsion is rendered degenerate by zero divisors, as each pure idempotent vector is destroyed by a nonzero scalar. Consequently, a regular-scalar concept is distinguished. Applications to algebraic operator models, invariant submodules, and bicomplex matrices are examined. Open problems pertain to the classification of constrained homomorphisms, homological invariants under additional structure, tensor products, and topological modules.

Keywords: bicomplex module; module homomorphism; idempotent decomposition; isomorphism theorem; exact sequence; free module; rank vector; zero divisor

Mathematical Notation and Symbols

Symbol	Meaning
$R=\mathscr{B}\mathbb{C}$	Bicomplex ring $\mathbb{C}\mathbf{e}_1\oplus\mathbb{C}\mathbf{e}_2$.
$M_j=\mathbf{e}_jM$	jth complex component of an R-module M.
Hom_R(M,N)	R-module homomorphisms from M to N.
ker φ, im φ	Kernel and image of φ.
M/A	Quotient module by a submodule A.
End_R(M), Aut_R(M)	Endomorphism ring and automorphism group.
$0\to A\to B\to C\to 0$	Short exact sequence.

Symbol	Meaning
$rk(M)=(r_1,r_2)$	Component rank vector when $r_j=\dim_{\mathbb{C}} M_j$.

1. Introduction

Module homomorphisms generalize linear maps from vector spaces to modules over rings. The bicomplex ring $\mathscr{B}\mathbb{C}$ contains the orthogonal central idempotents $\mathbf{e}_1$ and $\mathbf{e}_2$ and is isomorphic to $\mathbb{C}\times\mathbb{C}$. As a result, a bicomplex module is categorically equivalent to a pair of complex vector spaces. This observation simplifies proofs but also reveals phenomena hidden by field-based language: a cyclic module need not be free, rank may be a pair rather than a number, and ordinary torsion detects vectors supported in only one idempotent component.

The literature on bicomplex functional analysis often starts with topological modules and uses idempotent decomposition to establish Hahn–Banach, completeness, and operator results (Alpay et al., 2014; Luna-Elizarrarás et al., 2014; Kumar & Saini, 2016). The present paper isolates the underlying algebra. This is necessary because analytical statements about kernels, quotients, finite rank, and exact sequences depend first on their module-theoretic form.

The goals are to state the basic definitions without importing field assumptions; prove the principal isomorphism theorems; classify finite generation and freeness; analyze exact sequences and endomorphism rings; and explain zero-divisor torsion. The proofs are given directly and then interpreted componentwise, making clear which conclusions are general module facts and which are special to $\mathscr{B}\mathbb{C}$.

2. Bicomplex Modules and Idempotent Decomposition

Definition 2.1. A bicomplex module M is an abelian group together with scalar multiplication $\mathscr{B}\mathbb{C}\times M\to M$ satisfying the usual module axioms. A subset $A\subset M$ is a submodule when it is closed under addition, additive inverses, and bicomplex scalar multiplication.

Theorem 2.2. Idempotent decomposition. Every bicomplex module has the internal direct-sum representation

$$M=\mathbf{e}_1M\oplus\mathbf{e}_2M=M_1\mathbf{e}_1\oplus M_2\mathbf{e}_2, \qquad (2.1)$$

and $M_1=\mathbf{e}_1M$ and $M_2=\mathbf{e}_2M$ are complex vector spaces.

Proof. For $x\in M$, $x=(\mathbf{e}_1+\mathbf{e}_2)x=\mathbf{e}_1x+\mathbf{e}_2x$, so the two submodules span M. If y lies in their intersection, $y=\mathbf{e}_1u=\mathbf{e}_2v$. Multiplying by $\mathbf{e}_1$ gives $y=\mathbf{e}_1\mathbf{e}_2v=0$, so the sum is direct. Complex scalar multiplication on $\mathbf{e}_jM$ is defined through the jth coordinate of $\mathscr{B}\mathbb{C}$; the vector-space axioms follow from the module axioms. □

Definition 2.3. For a submodule $A\subset M$, write $A_j=\mathbf{e}_jA$. The quotient M/A consists of cosets x+A with the induced bicomplex operations.

Proposition 2.4. Every submodule and quotient decomposes componentwise:

$$A=A_1\mathbf{e}_1\oplus A_2\mathbf{e}_2, \quad M/A \cong (M_1/A_1)\mathbf{e}_1\oplus(M_2/A_2)\mathbf{e}_2. \qquad (2.2)$$

Proof. The decomposition of A is Theorem 2.2 applied to A. Define $\Psi(x+A)=(x_1+A_1)\mathbf{e}_1+(x_2+A_2)\mathbf{e}_2$. If $x-y\in A$, then $x_j-y_j\in A_j$, so Ψ is well defined. It is linear, surjective, and has zero kernel, hence is an isomorphism. □

Definition 2.5. The direct sum $M\oplus N$ is the module of pairs with componentwise operations. More generally, an internal direct sum $M=A\oplus B$ means $M=A+B$ and $A\cap B=\{0\}$.

3. Homomorphisms, Kernels, and Images

Definition 3.1. A bicomplex-module homomorphism $\varphi:M\to N$ satisfies $\varphi(x+y)=\varphi(x)+\varphi(y)$ and $\varphi(\alpha x)=\alpha\varphi(x)$ for all $x,y\in M$ and $\alpha\in\mathscr{B}\mathbb{C}$. It is a monomorphism, epimorphism, or isomorphism when it is injective, surjective, or bijective, respectively.

Proposition 3.2. For every homomorphism φ, ker φ is a submodule of M and im φ is a submodule of N.

Proof. If $x,y\in\ker\varphi$ and $\alpha,\beta\in\mathscr{B}\mathbb{C}$, then $\varphi(\alpha x+\beta y)=\alpha\varphi(x)+\beta\varphi(y)=0$. Thus the kernel is a submodule. If $u=\varphi(x)$, $v=\varphi(y)$, then $\alpha u+\beta v=\varphi(\alpha x+\beta y)$, so the image is a submodule. □

Definition 3.3. An endomorphism is a homomorphism $M\to M$; an automorphism is a bijective endomorphism. Their sets are denoted End_R(M) and Aut_R(M).

Theorem 3.4. Homomorphism decomposition. Every $\varphi\in\mathrm{Hom}\_R(M,N)$ is uniquely

$$\varphi=\varphi_1\mathbf{e}_1+\varphi_2\mathbf{e}_2,\quad \varphi(x_1\mathbf{e}_1+x_2\mathbf{e}_2)=\varphi_1(x_1)\mathbf{e}_1+\varphi_2(x_2)\mathbf{e}_2, \qquad (3.1)$$

with complex-linear $\varphi_j:M_j\to N_j$. Consequently,

$$\mathrm{Hom}\_R(M,N)\cong\mathrm{Hom}\_\mathbb{C}(M_1,N_1)\times\mathrm{Hom}\_\mathbb{C}(M_2,N_2). \qquad (3.2)$$

Proof. Linearity gives $\varphi(\mathbf{e}_j x)=\mathbf{e}_j\varphi(x)$, so φ preserves each component. Restriction defines φ_j, and complex linearity follows from the jth scalar coordinate. Conversely, a pair of complex-linear maps defines an R-linear map by Equation (3.1). The constructions are inverse and respect addition and composition. □

Corollary 3.5. A homomorphism is injective, surjective, or bijective if and only if both component maps have the corresponding property. Moreover,

$$\ker\varphi=(\ker\varphi_1)\mathbf{e}_1\oplus(\ker\varphi_2)\mathbf{e}_2,\qquad \mathrm{im}\,\varphi=(\mathrm{im}\,\varphi_1)\mathbf{e}_1\oplus(\mathrm{im}\,\varphi_2)\mathbf{e}_2. \qquad (3.3)$$

Proof. The identities follow by setting the two components of $\varphi(x)$ equal to zero and by describing all component images. The mapping properties follow from the identities. □

4. Isomorphism Theorems

Theorem 4.1. First isomorphism theorem. For $\varphi:M\to N$, the map

$$\Phi:M/\ker\varphi\to\mathrm{im}\,\varphi,\quad \Phi(x+\ker\varphi)=\varphi(x), \qquad (4.1)$$

is a well-defined bicomplex-module isomorphism.

Proof. If $x+\ker\varphi=y+\ker\varphi$, then $x-y\in\ker\varphi$ and $\varphi(x)=\varphi(y)$, so Φ is well defined. It is additive and R-linear because φ is. It is surjective by the definition of im φ. If $\Phi(x+\ker\varphi)=0$, then $\varphi(x)=0$, so $x\in\ker\varphi$ and the coset is zero; hence Φ is injective. Equivalently, component decomposition gives the two classical vector-space first isomorphism theorems, which recombine to the same map. □

Corollary 4.2. If φ is surjective, then $N\cong M/\ker\varphi$. If φ is injective, then $M\cong\mathrm{im}\,\varphi$.

Proof. These are the two special cases of Theorem 4.1. □

Theorem 4.3. Second isomorphism theorem. If A and B are submodules of M, then A+B is a submodule, $A\cap B$ is a submodule of A, and

$$(A+B)/B \cong A/(A\cap B). \qquad (4.2)$$

Proof. Define $\psi:A\to(A+B)/B$ by $\psi(a)=a+B$. It is R-linear. Every element of $(A+B)/B$ has a representative $a+b$ and equals $a+B$, so ψ is surjective. Its kernel is $\{a\in A:a\in B\}=A\cap B$. The first isomorphism theorem yields $A/(A\cap B)\cong(A+B)/B$. □

Theorem 4.4. Third isomorphism theorem. If $A\subset B\subset M$ are submodules, then B/A is a submodule of M/A and

$$(M/A)/(B/A) \cong M/B. \qquad (4.3)$$

Proof. Define $\theta:M/A\to M/B$ by $\theta(x+A)=x+B$. It is well defined because $A\subset B$. It is surjective and its kernel consists of $x+A$ with $x\in B$, namely B/A. Apply the first isomorphism theorem. □

Theorem 4.5. Correspondence theorem. For a fixed submodule $A\subset M$, the map $B\mapsto B/A$ is an inclusion-preserving bijection between submodules B of M containing A and submodules of M/A. It preserves sums, intersections, and quotients.

Proof. The inverse sends a submodule $C\subset M/A$ to its full preimage $\pi^{-1}(C)$ under the quotient map $\pi:M\to M/A$. Preimages are submodules containing $\ker \pi=A$, and $\pi^{-1}(B/A)=B$. Conversely, $\pi(\pi^{-1}(C))=C$ because π is surjective. Inclusion, sums, and intersections follow from image/preimage identities. For $A\subset B\subset C$, the third isomorphism theorem gives $(C/A)/(B/A)\cong C/B$. □

5. Exact Sequences and Splitting

Definition 5.1. A sequence of homomorphisms $A\to^{f}B\to^{g}C$ is exact at B when $\mathrm{im}\, f=\ker g$. A short exact sequence is $0\to A\to^{f}B\to^{g}C\to 0$. It splits if there is $s:C\to B$ with $g\circ s=I\_C$, equivalently $B\cong A\oplus C$ with the prescribed maps.

Proposition 5.2. A bicomplex sequence is exact if and only if both idempotent component sequences are exact complex vector-space sequences.

Proof. By Corollary 3.5, im f and ker g decompose as the images and kernels of the two component maps. Equality of bicomplex submodules is equivalent to equality in both components. □

Theorem 5.3. Every short exact sequence of bicomplex modules splits.

Proof. Decompose the short exact sequence into two short exact sequences of complex vector spaces. For each j choose a linear complement of $f_j(A_j)$ in B_j, or equivalently choose a linear right inverse $s_j:C_j\to B_j$ of g_j. Define $s=s_1\mathbf{e}_1+s_2\mathbf{e}_2$. Then $g\circ s=(g_1s_1)\mathbf{e}_1+(g_2s_2)\mathbf{e}_2=I\_C$. Thus $B=f(A)\oplus s(C)$. The choice is generally noncanonical. □

Corollary 5.4. Every bicomplex module is projective and injective as an algebraic $\mathscr{B}\mathbb{C}$-module; equivalently, the ring $\mathscr{B}\mathbb{C}\cong\mathbb{C}\times\mathbb{C}$ is semisimple.

Proof. The splitting of every short exact sequence yields the lifting and extension properties characterizing projective and injective modules. Alternatively, every module is a direct sum of vector spaces over two fields, and all vector-space exact sequences split. □

6. Free Modules, Finite Generation, and Rank

Definition 6.1. A module M is generated by $x_1,\ldots,x\_m$ if every $x\in M$ is $\Sigma_r\alpha_r x_r$. It is free of rank r if it has a basis and is isomorphic to R^r. Define the component rank vector $\mathrm{rk}(M)=(\dim\mathbb{C}M_1,\dim\mathbb{C}M_2)$, allowing infinite cardinals.

Theorem 6.2. M is finitely generated if and only if M_1 and M_2 are finite dimensional. If $r_j=\dim\mathbb{C}M_j$, the minimum number of generators is $\max(r_1,r_2)$.

Proof. If $x^1,\ldots,x^m$ generate M, their jth components span M_j, so $r_j\leq m$. Conversely, let $\{u_1,\ldots,u\_\{r_1\}\}$ and $\{v_1,\ldots,v\_\{r_2\}\}$ be component bases and set $m=\max(r_1,r_2)$. Pad the shorter list with zeros and define $x^k=u\_k\mathbf{e}_1+v\_k\mathbf{e}_2$. For any $x=x_1\mathbf{e}_1+x_2\mathbf{e}_2$, expand $x_1=\Sigma a\_ku\_k$ and $x_2=\Sigma b\_kv\_k$, then $x=\Sigma(a\_k\mathbf{e}_1+b\_k\mathbf{e}_2)x^k$. Hence m generators suffice, and the lower bound already proved makes m minimal. □

Theorem 6.3. A finite-dimensional bicomplex module M is free of rank r if and only if $\mathrm{rk}(M)=(r,r)$.

Proof. If $M\cong R^r$, then $\mathbf{e}_jM\cong\mathbb{C}^r$, so both dimensions are r. Conversely, choose bases $u_1,\ldots,u\_r$ and $v_1,\ldots,v\_r$ and define $x^k=u\_k\mathbf{e}_1+v\_k\mathbf{e}_2$. The spanning argument from Theorem 6.2 applies. If $\Sigma\alpha\_kx^k=0$, the two component basis expansions force every coefficient component to vanish, hence $\alpha\_k=0$. Thus $\{x^k\}$ is an R-basis. □

Example 6.4. The ideal $\mathbf{e}_1 R$ is cyclic and projective but not free as a nonzero R-module. Its rank vector is (1,0).

Proof. It is generated by $\mathbf{e}_1$ and is a direct summand of $R=\mathbf{e}_1 R \oplus \mathbf{e}_2 R$, so it is projective. If it were free of rank r, Theorem 6.3 would give rank vector (r,r), impossible for (1,0). □

Proposition 6.5. For finite rank vector (r_1,r_2),

$$\mathrm{End}_R(M) \cong M_{r_1}(\mathbb{C}) \times M_{r_2}(\mathbb{C}), \qquad (6.1)$$

$$\mathrm{Aut}_R(M) \cong GL_{r_1}(\mathbb{C}) \times GL_{r_2}(\mathbb{C}).$$

Proof. Apply the Hom decomposition with M=N and choose component bases. Composition corresponds to component matrix multiplication. An endomorphism is invertible exactly when both matrices are invertible. □

7. Torsion and Zero-Divisor Complications

Definition 7.1. Naive torsion is $\mathrm{Tor}(M)=\{x \in M : \alpha x=0 \text{ for some nonzero } \alpha \in R\}$. Regular torsion restricts α to regular elements, meaning non-zero-divisors.

Proposition 7.2. Every pure nonzero idempotent vector is naively torsion: if $x=\mathbf{e}_1 x_1 \neq 0$, then $\mathbf{e}_2 x=0$, and similarly with the indices reversed. By contrast, regular torsion is zero for every bicomplex module.

Proof. The first assertion uses $\mathbf{e}_1\mathbf{e}_2=0$ and the fact that $\mathbf{e}_2$ is nonzero. A regular element $\alpha=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2$ has both coefficients nonzero and is therefore a unit. If $\alpha x=0$, multiplying by α^{-1} gives $x=0$. □

Remark 7.3. Thus the field-based statement "torsion-free" is ambiguous over $\mathscr{B}\mathbb{C}$. Under naive torsion, a module with any nonzero pure component has torsion; under regular torsion, every module is torsion-free. Component rank and annihilator ideals are more informative invariants.

7.4 Ideals, Tensor Products, and Categorical Form

Theorem 7.4. The only ideals of the bicomplex ring R are {0}, $\mathbf{e}_1 R$, $\mathbf{e}_2 R$, and R.

Proof. Under $R \cong \mathbb{C}\times\mathbb{C}$, an ideal I projects to ideals I_1 and I_2 of the fields $\mathbb{C}$. Each projection is either {0} or $\mathbb{C}$. Therefore I is one of the four products $\{0\}\times\{0\}$, $\mathbb{C}\times\{0\}$, $\{0\}\times\mathbb{C}$, or $\mathbb{C}\times\mathbb{C}$, corresponding to the listed ideals. □

Corollary 7.5. For $x=x_1\mathbf{e}_1+x_2\mathbf{e}_2$, the annihilator ann(x) is {0} when both components are nonzero, $\mathbf{e}_2 R$ when $x_1 \neq 0$ and $x_2=0$, $\mathbf{e}_1 R$ when $x_1=0$ and $x_2 \neq 0$, and R when x=0.

Proof. A scalar α annihilates x exactly when $\alpha_1 x_1=0$ and $\alpha_2 x_2=0$. Since each component scalar field has no zero divisors, the four cases follow. □

Theorem 7.6. Submodule-lattice product. The map $A \mapsto (A_1,A_2)$ is a lattice isomorphism between submodules of M and pairs of complex subspaces $A_j \subset M_j$. It preserves arbitrary sums and intersections.

Proof. Every submodule decomposes as $A=A_1\mathbf{e}_1 \oplus A_2\mathbf{e}_2$. Conversely, any pair of complex subspaces defines a submodule. Inclusion is componentwise. Scalar multiplication by central idempotents shows that sums and intersections decompose into the corresponding component operations. □

Theorem 7.7. Tensor-product decomposition. For bicomplex modules M and N,

$$M \otimes_R N \cong (M_1 \otimes_{\mathbb{C}} N_1)\mathbf{e}_1 \oplus (M_2 \otimes_{\mathbb{C}} N_2)\mathbf{e}_2. \qquad (7.1)$$

Proof. Mixed tensors vanish: for $m_1 \in M_1$ and $n_2 \in N_2$, $m_1 \otimes n_2=\mathbf{e}_1 m_1 \otimes n_2 = m_1 \otimes \mathbf{e}_1 n_2=0$. Thus only equal-idempotent components remain. Define $\Psi(m\otimes n)=(m_1\otimes n_1)\mathbf{e}_1+(m_2\otimes n_2)\mathbf{e}_2$. The balanced relations make Ψ well defined. The inverse sends a pure component tensor to the same elementary R-tensor. The two maps agree on generators and are inverse. □

Corollary 7.8. If M and N have finite rank vectors (r_1,r_2) and (s_1,s_2), then $M \otimes_R N$ has rank vector $(r_1 s_1, r_2 s_2)$. If both are free of ranks r and s, their tensor product is free of rank rs.

Proof. Complex tensor dimensions multiply. Equality of the two resulting dimensions is preserved in the free case. □

Proposition 7.9. Matrix representation. A homomorphism $\varphi:R^m\to R^n$ is represented uniquely by $A=A_1\mathbf{e}_1+A_2\mathbf{e}_2$ with $A_j\in M_{n\times m}(\mathbb{C})$. When $m=n$, φ is an automorphism if and only if $\det A_1$ and $\det A_2$ are nonzero.

Proof. Choose the standard free bases and use the Hom decomposition. A square matrix over R has an inverse exactly when both complex component matrices do; this is equivalent to the two determinant conditions. A determinant lying in the null cone signals singularity in at least one component. □

Example 7.10. Consider $A=\mathrm{diag}(1,\mathbf{e}_1)$ on R^2. Its component matrices are $A_1=I_2$ and $A_2=\mathrm{diag}(1,0)$. Hence $\ker A=R\cdot(0,\mathbf{e}_2)$, the range has rank vector (2,1), and A is not invertible even though one component matrix is the identity.

Proof. The second coordinate is multiplied by $\mathbf{e}_1$, so its pure $\mathbf{e}_2$ part is killed. The component kernel and range formulas give the stated modules. Since $\det A=\mathbf{e}_1$ is a zero divisor, no inverse exists. □

Theorem 7.11. Categorical equivalence. The category of bicomplex modules and bicomplex homomorphisms is equivalent to the product category $\mathrm{Vect}\mathbb{C}\times\mathrm{Vect}\mathbb{C}$.

Proof. Define $F(M)=(M_1,M_2)$ and $F(\varphi)=(\varphi_1,\varphi_2)$. Define $G(V,W)=V\mathbf{e}_1\oplus W\mathbf{e}_2$ and let G act componentwise on maps. The natural maps $x\mapsto(\mathbf{e}_1x,\mathbf{e}_2x)$ and $(v,w)\mapsto v\mathbf{e}_1+w\mathbf{e}_2$ give natural isomorphisms $GF\cong\mathrm{Id}$ and $FG\cong\mathrm{Id}$. □

The categorical statement explains several earlier results at once: all algebraic exact sequences split because vector-space sequences split; projective and injective modules are automatic; and tensor products and Hom spaces are computed componentwise. It also identifies where nontrivial theory must enter. Topology restricts allowable splittings to continuous maps, inner products restrict maps to adjointable ones, and positivity or tensor norms can impose relations not present in the bare product category.

Proposition 7.12. For finite modules M and N, the rank vector of $\mathrm{Hom}_R(M,N)$ is (r_1s_1,r_2s_2) when $\mathrm{rk}(M)=(r_1,r_2)$ and $\mathrm{rk}(N)=(s_1,s_2)$.

Proof. The complex space $\mathrm{Hom}\mathbb{C}(M_j,N_j)$ has dimension r_js_j. Apply Equation (3.2). □

A short exact sequence can be split concretely by choosing component bases. For example, let $B=R^3$, let A be generated by (1,0,0) and $(0,\mathbf{e}_1,0)$, and form $C=B/A$. The component dimensions are $\mathrm{rk}(A)=(2,1)$, $\mathrm{rk}(B)=(3,3)$, and $\mathrm{rk}(C)=(1,2)$. Choosing complementary component basis vectors constructs a section $C\to B$, but no preferred section exists without extra geometry.

This example also shows why a single "dimension" can violate additivity. Rank vectors satisfy $\mathrm{rk}(B)=\mathrm{rk}(A)+\mathrm{rk}(C)$ componentwise, while the minimum number of generators obeys $\max(3,3)=3$, $\max(2,1)=2$, and $\max(1,2)=2$, which is not additive. Module invariants should therefore be chosen for the operation under study.

7.13 Algebraic Duality

Theorem 7.13. For a finite-dimensional bicomplex module M with rank vector (r_1,r_2), the algebraic dual $M=$*$\mathrm{Hom}_R(M,R)$ has the same rank vector, and the canonical evaluation map $J:M\to M$**, $J(x)(f)=f(x)$, is an isomorphism.

Proof. By the Hom decomposition, $M=$*$\mathrm{Hom}\mathbb{C}(M_1,\mathbb{C})\mathbf{e}_1\oplus\mathrm{Hom}\mathbb{C}(M_2,\mathbb{C})\mathbf{e}_2$, whose component dimensions are r_1 and r_2. Applying the same argument again gives M** with the original components. The component evaluation maps are the standard finite-dimensional vector-space isomorphisms, so their pair J is an R-isomorphism. □

Corollary 7.14. If M is free of rank r, then M^* is free of rank r. If M has unequal component ranks, dualization preserves that inequality and does not make the module free.

Proof. Apply Theorem 6.3 and the rank-vector identity in Theorem 7.13. □

Algebraic reflexivity here is finite-dimensional and should not be confused with Banach-space reflexivity. In topological modules, the continuous dual may be smaller than the algebraic dual, and the canonical map into the continuous bidual is onto exactly when both component Banach spaces are reflexive.

8. Applications and Interpretive Analysis

Bicomplex matrices are endomorphisms of free modules and can be decomposed into paired complex matrices. Kernels, images, Jordan forms, and invariant submodules can be computed componentwise; however, modules of unequal component rank naturally originate as invariant summands. This structure is employed by Johnston and Wahl (2023) to analyse invariant-subspace lattices and bicomplex matrices.

The first isomorphism theorem in operator theory algebraically identifies M/ker T with ran T; however, a topology must be incorporated before an isomorphism of normed modules can be claimed. Exact-sequence splitting elucidates the simplicity of purely algebraic extension problems over $\mathscr{B}\mathbb{C}$, whereas bounded or continuous splitting may fail in the absence of complemented component ranges.

In algebraic models of paired channels, a homomorphism $\varphi=\varphi_1 e_1+\varphi_2 e_2$ can have varying ranks and nullities. A singular determinant or rank number can therefore be misleading. A more precise description is provided by the rank vector, annihilator, and component automorphism groups. Explicit component and positivity data are also necessary for tensor constructions and quantum-information maps.

9. Open Problems and Limitations

Topological enquiries are not resolved by algebraic semisimplicity. A bounded projection onto a closed submodule of a bicomplex Banach module is only possible when both component subspaces are complemented, despite the fact that it is componentwise closed. In the same vein, algebraic right inverses are not required to be continuous.

Additional research may investigate topological exact categories, completed tensor products, bicomplex operator modules, graded and involutive structures, and homomorphisms that preserve inner products or positive cones. Although the ring is semisimple, ordinary homological invariants vanish for unrestricted $\mathscr{B}\mathbb{C}$-modules. However, nontrivial extension theories can be produced by additional analytic, order, or symmetry constraints. Rank vectors should be preserved in computational classifications, rather than being collapsed into a single scalar.

10. In summary

Bicomplex-module homomorphisms are precisely pairs of complex-linear mappings. This principle provides component formulations for kernels, images, quotients, direct sums, exactness, endomorphisms, and automorphisms, and it facilitates comprehensive proofs of the correspondence theorem and the three isomorphism theorems. Modules of unequal component dimensions are projective without being free, despite the fact that every algebraic brief exact sequence divides. Finite generation is regulated by two dimensions: the minimum number of generators is the maximum number, and equality is necessary for freeness. Annihilators and rank vectors are more advantageous invariants, as zero divisors render naive torsion ubiquitous and regular torsion trivial. The algebraic foundation for bicomplex multi-normed and Hilbert-module operator theory is established by these conclusions.

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