# An EOQ Model with Non-homogeneous Poisson Demand Process

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**Abstract:**
Classical EOQ models with exponentially increasing demand assume deterministic and continuous demand D(t) = a$e^{bt}$. In reality, for products in growth phase, customer arrivals are discrete, random, and the arrival rate grows with time due to advertising and word-of-mouth effect. Homogeneous Poisson Process with constant rate λ fails to capture this growth. The present study proposes an EOQ model with Non-Homogeneous Poisson Process (NHPP) where arrival intensity is λ (t) = a$e^{bt}$, a > 0, b > 0. In this paper the demand is assumed to be stochastic in nature. To handle the time dependent demand process, generally there are two approaches namely (a) Treating the demand process on continuous time and continuous state Markov process usually known as Weiner process, and (b) The discrete Poisson process with time dependent parameter can be used. In this paper the latter approach is used in modelling time dependent demand.

**Key Words:** Inventory model, Stochastic demand, Non-Homogeneous Poisson Process, EOQ model, Inventory system.

**Introduction:**
Traditional inventory models assume Poisson demand with constant mean. In practice - retail, spare parts, traffic, call centers - demand rate fluctuates with time of day, day-of-week, and season. In classical inventory theory, demand N(t) is assumed to be Poisson with constant rate λ. This is unrealistic. A restaurant in Vijayawada sees 5 orders/hour at 9am, 25 orders/hour at 12:30pm, and 17 orders/hour at 8pm. This requires λ(t).
The NHPP is defined by: P (N (t + h) − N(t) = 1) ≈ λ(t)h,
P (N (t + h) − N(t) > 1) = o (h).
Hadley and Whitin (1979) have discussed a simple EOQ model when the demand follows Poisson distribution. They assumed that each calling unit (customer) demands exactly one unit and the inventory system is viewed as a single server Markovian queue. If the parameter λ is constant in Poisson process then it is known as Homogeneous Poisson process. Sarma (1983) was first to introduce two-warehouse model, but with deterministic demand. He assumed Rented Warehouse (RW) holding cost > Own Warehouse (OW). Rao et al. (2005) first used λ (t) = a$e^{bt}$ in inventory for electronic goods. They proved expected total cost is convex in T. Gupta and Singh (2011) developed EOQ with Non-Homogeneous Poisson Process (NHPP) and exponentially increasing demand, showed 12-15% cost saving over deterministic model when b > 0.1. Sana (2012) included deterioration with NHPP, showed θr > θo increases expected RW empty time variance. Chen et al. (2020) used NHPP with λ (t) = a$e^{bt}$ for COVID vaccine inventory with two cold

storages. Singh et al. (2025) have worked on sustainable two-warehouse with NHPP and carbon emission cost.

A brief note on some important results of Poisson process given by Medhi (1982) is given below.

**Important Results on Homogeneous Poisson Process**

Let N (t) is the number of occurrences in the interval (0, t). The Poisson law for the stochastic process {N (t)} can be defined with the following postulates.

**i. Independence:** N (t) is independent of the number of occurrences in an interval prior to the interval (0, t), i.e., future changes in N (t) are independent of the past changes.

**ii. Homogeneity in Time:** $P_n(t)$ depends only on the length of the interval and is independent of where this interval is situated, i.e., $P_n(t)$ gives the probability of the number of occurrences in the interval ($t_1$, t + $t_1$) for every $t_1$.

**iii. Regularity:** In an interval of infinitesimal length 'h', the probability of exactly one occurrence is λh + o (h) and that of more than one occurrence is of O (h). Under the postulates (i), (ii), and (iii), N (t) follows Poisson distribution with mean λt, i.e., $P_n$ (t) is given by Poisson law:

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \text{ n = 0,1, 2, . . . .} \qquad \text{. . . . . (1)}$$

The second postulate implies that the Poisson process is homogeneous in time. The time between two successive events follows exponential distribution with parameter λ. That means the inter arrival time 'x' follows an exponential distribution. The waiting time up to $n^{th}$ demand will have a Gamma density and is given by

$$g(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)}, \text{ x > 0.} \qquad \text{. . . . . (2)}$$

This result can be interpreted in the inventory theory as follows:

Let Q be the number of units available at time t = 0 and let the demand follows the Poisson process with mean λ. Here, the assumption is that the demand is only one unit intern means that exactly Q demands will occur to deplish the inventory when shortages are not allowed.

One can use Renewal theoretic arguments in case of non-Poisson demands in the inventory models. It is also possible that λ is a random variable and if it follows Geometric distribution, then it is called Stuttering Poisson Process (Hadley and Whitin [49]). When the parameter λ is influenced by time then the corresponding process is known as Non-homogeneous Poisson process. This process is elaborated hereunder.

**Non-homogeneous Poisson Process or Time Dependent Poisson Process**

In this case λ is assumed to be non-random function of time such that

$$\begin{aligned} P\{N(h) = k / N(t) = n\} &= \lambda(t)\,h + o(h), && k \geq 1 \\ &= o(h), && k \geq 2 \qquad \text{. . . . . (3)} \\ &= 1 - \lambda(t)\,h + o(h), && k \geq 0 \end{aligned}$$

Here the postulate (ii) of the classical Poisson process regarding homogeneity does not hold to and this process is called a non-homogeneous Poisson process. For this process the p.g.f is given by

$$Q(s,t) = e^{\{m(t)(s-1)\}} \qquad \text{. . . . . (4)}$$

where

$$m(t) = \int_0^t \lambda(x)dx \text{ , is the expectation of N (t)} \qquad \text{. . . . . (5)}$$

Thus, in the non-homogeneous Poisson process the mean number of occurrences between (0, t) is m (t). If we take $\lambda(t) = \lambda t$ , we get $m(t) = \frac{\lambda t^2}{2}$ and obviously if $\lambda(t) = \lambda$ , we get $m(t) = \lambda t$ . This is the mean for the homogeneous Poisson process. Another important result can be noted that the probability of having exactly Q events in the interval (0, t) is given by

$$p_n(t) = \frac{[m(t)]^n e^{-m(t)}}{n!} \qquad \text{. . . . . (6)}$$

The above probability holds good for the interval [0, m (t)]. From the above $p_n(t)$ , one can obtain homogeneous Poisson process results when $\lambda(t) = \lambda$ . We make use of the following results due to Renewal theorem without proof for our further discussion.

**Result (i):** In inventory context the cycle length when the demand follows non-homogeneous Poisson process is given by

$$g(t) = \frac{e^{\frac{-\lambda t^2}{2}} \left( \frac{\lambda t^2}{2} \right)^{Q-1}}{\Gamma Q} \lambda t \qquad \text{. . . . . (7)}$$

**Result (ii):** The expected cycle length is given by

$$E(T) = \left( \frac{2}{\lambda} \right)^{\frac{1}{2}} \frac{\Gamma(Q + \frac{1}{2})}{\Gamma Q} \qquad \text{. . . . . (8)}$$

For the computational purposes the following approximation is used wherever necessary

$$\frac{\Gamma(Q + \frac{1}{2})}{\Gamma Q} \cong \sqrt{Q} \qquad \text{. . . . . (9)}$$

With this discussion the following lot size model with the demand following non-homogeneous Poisson process is established.

**Economic Lot-size Model with Non-homogeneous Poisson Process**

The following assumptions are underlying for establishing the EOQ model with non-homogeneous Poisson process.

(i) Shortages are not allowed
(ii) The lead time is zero
(iii) The demand is generated by the non-linear time dependent Poisson process such that $\lambda(t) = \lambda e^{\alpha t}$ , $0 < \alpha < 1$
(iv) Holding cost is H per unit per unit time and the set-up cost is Rs. $C_3$ per order.
(v) The planning horizon is finite of length U.

The problem is to determine the optimal lot size during the planning period U by balancing the associated costs. Obviously, Q, the quantity ordered is an integer.

**Mathematical Formulation of the Model:**

At first, we note that the time to deplete all the 'Q' units is the random variable T. A second order is placed only after this T. For the given U, the policy is that all Q units demanded should be cleared before U. Since the demand is random it may possible that T > U. In this case whenever T > U, define θ as the unit penalty cost of violating the horizon U in a single order.

Let the probability that the cycle length exceeds U is defined as $\beta = \int_U^\infty g(t)dt$. This means that the probability of violating U in a single order is β. The complementary event is that the probability of non-violating U is (1 - β).

The main objective is to workout the inventory cost in each case and multiply them with appropriate probability either β or (1 - β).

**Expected Cost Function of the Model:**

The costs associated with the inventory system under consideration are only holding cost and set-up cost. When T ≤ U, the expected cost of holding is given by

$$B_1 = H\int_0^\infty \left\{ \int_0^t x\lambda e^{\alpha x} dx \right\} g(t)dt \qquad \text{.....(10)}$$

and this will happen with probability β. Suppose T > U, then there is a cost of holding the inventory and that of violating the time horizon. The expected cost of holding in this case is given by

$$B_2 = H\int_0^\infty \left\{ \int_0^U x\lambda e^{\alpha x} dx \right\} g(t)dt \qquad \text{.....(11)}$$

The expected cost of violation is given by

$$B_3 = J\int_0^\infty \left\{ \int_U^t x\lambda (e^{\alpha x} - U) dx \right\} g(t)dt \qquad \text{.....(12)}$$

where J = (H + θ)

The total expected cost of the inventory system in a cycle is given by

$$E(C) = C_3 + (1-\beta)B_1 + \beta(B_2 + B_3) \qquad \text{.....(13)}$$

It can be noted that the quantities $B_1$, $B_2$, and $B_3$ are the functions of Q as hereunder.

$$B_1 = H\int_0^\infty \left\{ \int_0^t x\lambda e^{\alpha x} dx \right\} g(t)dt$$

Taking first order approximation for $e^x$, we get

$$B_1 = H\left[\int_0^\infty \lambda^2 t^2 \frac{e^{\frac{-\lambda t^2}{2}}\left(\frac{\lambda t^2}{2}\right)^{Q-1}}{\Gamma Q}dt + \alpha\int_0^\infty \lambda^2 t^3 \frac{e^{\frac{-\lambda t^2}{2}}\left(\frac{\lambda t^2}{2}\right)^{Q-1}}{\Gamma Q}dt\right]$$

Let $\frac{\lambda t^2}{2} = P \Rightarrow t = \sqrt{\frac{2P}{\lambda}}, dt = \frac{dP}{\lambda}$ so that $B_1$ reduces to

$$B_1 = 2H\left[\int_0^\infty \frac{e^{-P}P^{Q-1+1}}{\Gamma Q}dP + \alpha\sqrt{\frac{2}{\lambda}}\int_0^\infty \frac{e^{-P}P^{Q-1+\frac{3}{2}}}{\Gamma Q}dP\right]$$

$$\therefore B_1 = 2H\left[\frac{\Gamma(Q+1)}{\Gamma Q} + \alpha\sqrt{\frac{2}{\lambda}}\frac{\Gamma(Q+\frac{3}{2})}{\Gamma Q}\right] \qquad \text{.....(14)}$$

Similarly

$$B_2 = H\int_0^\infty\left\{\int_0^U x\lambda e^{\alpha x}dx\right\}g(t)dt$$

Taking first order approximation for $e^x$ and simplifying, we get

$$B_2 = H\frac{\lambda}{\alpha^2}\int_0^\infty \left\{e^{\alpha U}(\alpha U - 1) + 1\right\}g(t)dt$$

$$= H\lambda^2 U^2\left[\int_0^\infty \frac{e^{\frac{-\lambda t^2}{2}}\left(\frac{\lambda t^2}{2}\right)^{Q-1}}{\Gamma Q}dt + \alpha\int_0^\infty t\frac{e^{\frac{-\lambda t^2}{2}}\left(\frac{\lambda t^2}{2}\right)^{Q-1}}{\Gamma Q}dt\right]$$

$$\therefore B_2 = H\left[\lambda^2 U^2 + (\alpha U^2\sqrt{2\lambda})\frac{\Gamma(Q+\frac{1}{2})}{\Gamma Q}\right] \qquad \text{.....(15)}$$

and

$$B_3 = J\int_0^\infty\left\{\int_U^t x\lambda(e^{\alpha x} - U)dx\right\}g(t)dt$$

$$B_3 = J\lambda\int_0^\infty\left[\frac{e^{\alpha t}}{\alpha^2}(\alpha t - 1) + \frac{e^{\alpha U}}{\alpha^2}(1 - \alpha U) - \frac{U}{2}(t^2 - U^2)\right]g(t)dt$$

Using first order approximation for $e^x$ and simplifying, we get

$$B_3 = \frac{J(2-U)}{2}\int_0^\infty \lambda(t^2 - U^2)g(t)dt$$

$$= \frac{J(2-U)}{2}\left[\int_0^\infty \lambda t^2 \frac{e^{\frac{-\lambda t^2}{2}}\left(\frac{\lambda t^2}{2}\right)^{Q-1}}{\Gamma Q}\lambda e^{\alpha t}dt - \lambda U^2\int_0^\infty g(t)dt\right]$$

$$\therefore B_3 = (H+\theta)(2-U)\left[\frac{\Gamma(Q+1)}{\Gamma Q} + \alpha\sqrt{\frac{2}{\lambda}}\frac{\Gamma(Q+\frac{3}{2})}{\Gamma Q} - \lambda U^2\right] \qquad .....(16)$$

The expected cost as a function of Q becomes

$$E(C) = C_3 + (1-\beta)2H\left[\frac{\Gamma(Q+1)}{\Gamma Q} + \alpha\sqrt{\frac{2}{\lambda}}\frac{\Gamma(Q+\frac{3}{2})}{\Gamma Q}\right] +$$

$$\beta\left[HU^2[\lambda^2 + \alpha\sqrt{2\lambda Q}] + (H+\theta)(2-U)\left\{\frac{\Gamma(Q+1)}{\Gamma Q} + \alpha\sqrt{\frac{2}{\lambda}}\frac{\Gamma(Q+\frac{3}{2})}{\Gamma Q} - \lambda U^2\right\}\right]$$

$$.....(17)$$

The expected duration of the cycle length is given by

$$E(T) = \int_0^\infty t g(t)dt \qquad .....(18)$$

$$E(T) = \int_0^\infty t\frac{e^{\frac{-\lambda t^2}{2}}\left(\frac{\lambda t^2}{2}\right)^{Q-1}}{\Gamma Q}\lambda e^{\alpha t}dt$$

Taking first order approximation for $e^{\alpha t}$ and simplifying, we get

$$E(T) = \sqrt{\frac{2}{\lambda}}\frac{\Gamma(Q+\frac{1}{2})}{\Gamma Q} + \frac{2\alpha}{\lambda}\frac{\Gamma(Q+1)}{\Gamma Q} \qquad .....(19)$$

Using the approximations

$\frac{\Gamma(Q+\frac{1}{2})}{\Gamma Q} = \sqrt{Q}, \frac{\Gamma(Q+1)}{\Gamma Q} = Q and \frac{\Gamma(Q+\frac{3}{2})}{\Gamma Q} = Q + \frac{1}{2}$, we get

$$\therefore E(T)=\sqrt{\frac{2Q}{\lambda}}+\frac{2\alpha Q}{\lambda} \qquad \text{.....}(20)$$

Hence the expected cost of the entire inventory system per unit time is given by

$$C(Q)=\frac{E(C)}{E(T)}$$

$$C(Q)=\frac{C_3+2H[Q+\alpha\sqrt{\frac{2}{\lambda}}(Q+\frac{1}{2})]+\beta HU^2\lambda^2+\alpha\beta HU^2\sqrt{2\lambda Q}-2H\beta\lambda U^2}{\left(\sqrt{\frac{2Q}{\lambda}}+\frac{2\alpha Q}{\lambda}\right)}$$

$$+\frac{\beta\theta(2-U)\left\{Q+\alpha\sqrt{\frac{2}{\lambda}}(Q+\frac{1}{2})-\lambda U^2\right\}}{\left(\sqrt{\frac{2Q}{\lambda}}+\frac{2\alpha Q}{\lambda}\right)} \qquad \text{.....}(21)$$

Since Q is non-negative integer, the necessary condition for C (Q*) to be minimum at $Q^*=Q_0^*$ is

$$\Delta C(Q_0^*-1)\le 0\le \Delta C(Q_0^*), \qquad \text{.....}(22)$$

where $\Delta C(Q_0^*)=C(Q_0^*+1)-C(Q_0^*)$ .....(23)

Solving the above equations one can obtain Q* of Q. The corresponding cost can be found from (21). However, we can find the continuous optimum value of Q by differentiating C (Q) with respect to Q and equating to zero by ensuring that the second derivative is greater than zero. We denote this optimal quantity of Q as Q*.

i.e., $\frac{dC(Q)}{dQ}=0\Rightarrow$

$$\left(\frac{1}{\sqrt{2\lambda Q}}+\frac{2\alpha}{Q}\right)\left[C_3+\alpha H\sqrt{\frac{2}{\lambda}}+\beta HU^2\lambda^2+2H\beta\lambda U^2+\alpha\beta\theta(2-U)/(\sqrt{2\lambda})+\beta\theta\lambda U^2(2-U)\right]$$

$$-\left(\sqrt{\frac{2Q}{\lambda}}-\sqrt{\frac{Q}{2\lambda}}\right)\left[2H+2\alpha H\sqrt{\frac{2}{\lambda}}+\beta\theta\lambda(2-U)+\alpha\beta\theta(2-U)\sqrt{\frac{2}{\lambda}}\right]+\alpha^2\beta HU^2\sqrt{\frac{2Q}{\lambda}}=0$$

.....(24)

The true integer optimum value can be obtained by evaluating C (Q) in the neighbourhood (Q*)

**Sensitivity Analysis of the model:**

The working of the model is illustrated with the following parametric values expressed in appropriate units.
[$C_3$, H, λ, U, β, α, θ] = [100, 2, 10, 1, 0.2, 0.05, and 0.01]

The continuous solution of Q using (23) is found to be Q* = 45.00881 units and the minimum cost is Rs. 94.57329. The expected cycle length in this case is 3.45038 which mean; with this order quantity, the average cycle length exceeds the horizon. To determine the exact integer solution, the cost function is evaluated in the neighbourhood of Q* and the true optimum is found to be Q*= 42 and the optimum cost is Rs. 94.51. The expected cycle length is 3.318275. The behaviour of the cost function is shown in the following figure:

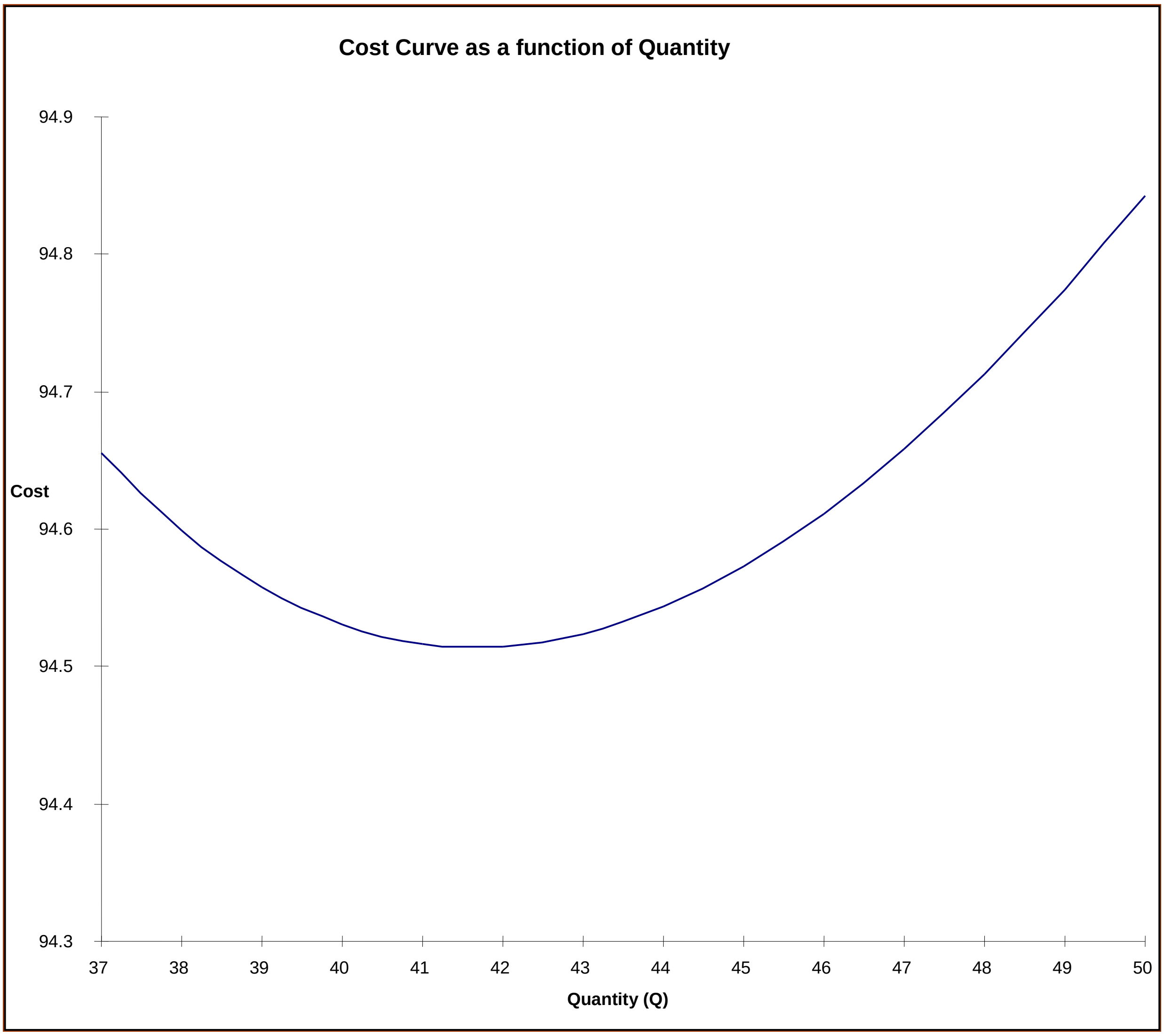


Figure 1

The sensitivity of the model with respect to changes in λ has been studied. For increasing values of λ the values of Q*, E (T) and the cost are shown in table below.

**Table 1: Variations in λ**

| λ | Q* | E(T) | Optimal Cost |
|---|---|---|---|
| 5 | 37.5446 | 4.6262 | 57 |
| 10 | 45.0088 | 3.4504 | 95 |
| 15 | 59.3935 | 3.2101 | 139 |
| 20 | 79.5395 | 3.2179 | 192 |
| 25 | 105.0714 | 3.3196 | 256 |
| 30 | 135.7519 | 3.4608 | 328 |
| 35 | 171.3795 | 3.6191 | 410 |
| 40 | 211.7650 | 3.7834 | 501 |
| 45 | 256.7253 | 3.9484 | 602 |
| 50 | 306.0824 | 3.4112 | 710 |

**Observations:**

➢ The EOQ increases with λ but there is a marginal increase in the duration between successive orders as λ increases.

➢ Since the demand is time–dependent in nature, the rate of demand increases as time increases. This can be demonstrated by using large values of λ. The associated cost also increases linearly with λ and is shown in figure 2.

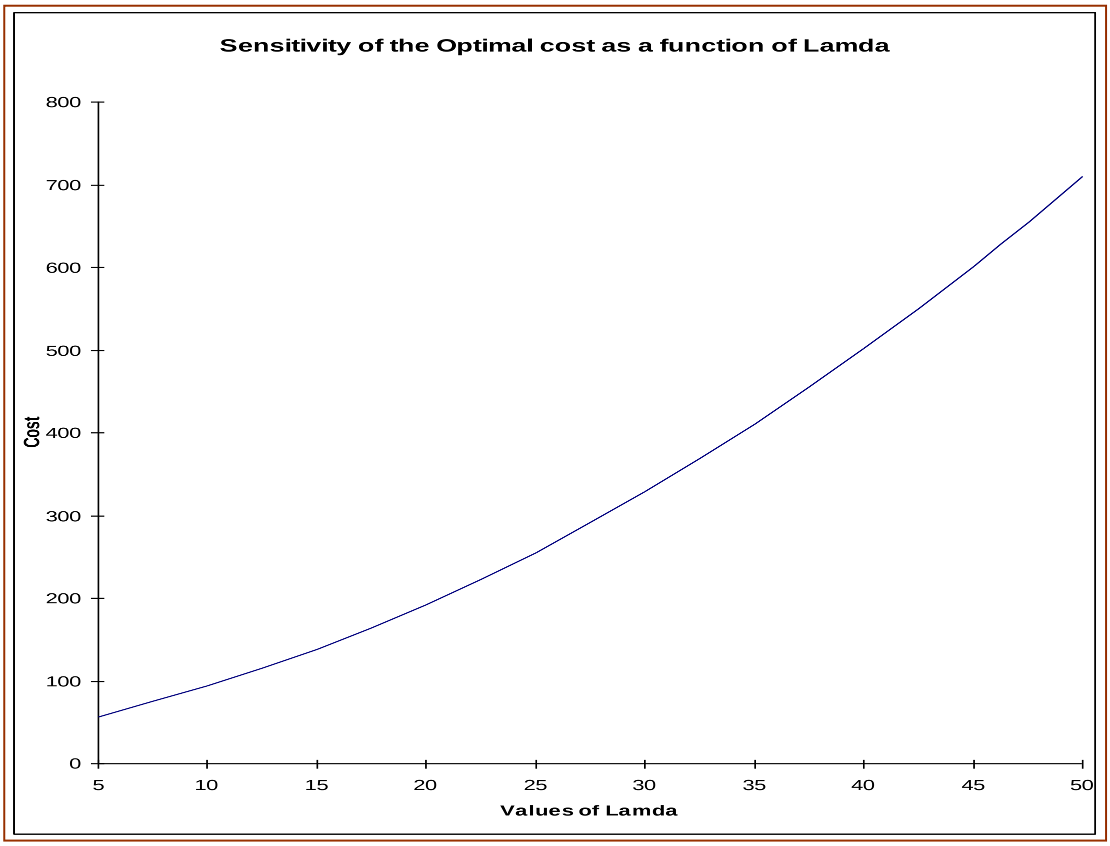


Figure 2

The variation in the length of horizon U on the model is of interest. Since the penalty cost associated with the large values of U is less than the penalty cost associated with the least values of time-horizon U. Hence the behaviour of the model with respect to variations in U has been portrayed in the following table and the nature of the cost function with respect to changes in U is depicted in figure 3

**Table 2: Variations in U**

| U | Q* | E(T) | Optimal Cost |
|---|---|---|---|
| 1 | 45.0088 | 3.4504 | 95 |
| 2 | 92.5266 | 5.2270 | 104 |
| 3 | 160.3567 | 7.2629 | 124 |
| 4 | 255.4856 | 9.7031 | 145 |
| 5 | 364.5266 | 12.1837 | 165 |
| 6 | 487.6148 | 14.7515 | 183 |
| 7 | 619.0859 | 17.3182 | 199 |
| 8 | 753.7519 | 19.8156 | 216 |
| 9 | 887.5094 | 22.1981 | 231 |
| 10 | 1017.1710 | 24.4347 | 246 |

It is observed from the above table that the increase in U i.e., the length of the horizon results the increase in EOQ and also increase in cost of the system.

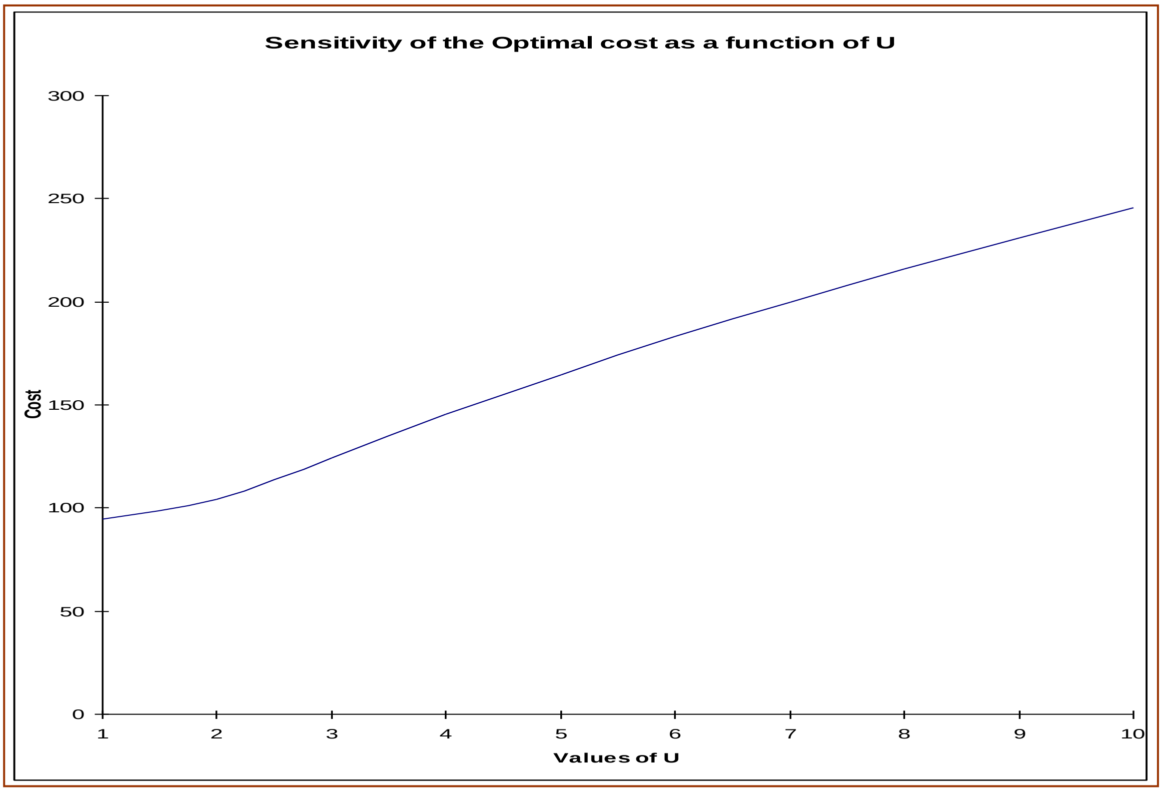


Figure 3

**Conclusion:**

In this paper we considered the concept of non-homogeneous Poisson process to study the effect of time dependent demand using non-linear realisation for the demand function. We have derived expressions for the EOQ and the expected duration between the successive orders. The model would be more challenging in case of multiple orders with this type of demand process. However, the static models in which the demand variation is known to behave as a random variable with a specified probability distribution having a fixed planning period will be mathematically complex in nature, itself since it is very difficult to obtain a closed form solution for the phenomenon under consideration.

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