Some Applications of Fractional Calclus Operator

Dr. Bonala Madhavi

Assistant Professor of Mathematics
Pingle Government College for Women, (Autonomous)
Hanmakonda – 506370, Telangana, India

Abstract:
The aim of this paper is to discuss a new subclass $TS(\hbar, \sigma, \varsigma, \vartheta)$ of univalent functions with negative coefficients related to fractional calclus operator in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. We obtain basic properties like coefficient inequality, distortion and covering theorem, radii of starlikeness, convexity and close-to-convexity, extreme points, Hadamard product, and closure theorems for functions belonging to our class.

2020 Mathematical Sciences Classification. 30C45.

Keywords and phrases. univalent , fractional operator , starlike, extreme points, Hadamard product.

1. Introduction

Let A denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disc $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in $\mathbb{U}$. A function $u \in A$ is a starlike function of the order $m, 0 \leq m < 1$, if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > m, z \in \mathbb{U}. \tag{1.2}$$

We denote this class with $S^*(m)$.
A function $u \in A$ is a convex function of the order $m, 0 \leq m < 1$, if it fulfils

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > m, z \in \mathbb{U}. \tag{1.3}$$

We denote this class with $K(m)$.
Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in $\mathbb{U}$ respectively.
Let T denotes the class of functions analytic in $\mathbb{U}$ that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \; z \in, \mathbb{U} \tag{1.4}$$

and let $T^*(m) = T \cap S^*(m)$, $C(m) = T \cap K(m)$. The class $T^*(m)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [11].
Fractional calculus is used in many investigations lately due to its numerous applications in diverse fields of research. Recently published thorough review articles ([1], [7]) discuss the history of fractional calculus and provide references to its many applications in science and engineering.
Since the connection between hypergeometric functions and the theory of univalent functions was established in 1985 with the proof of Bieberbach's conjecture [2], the confluent hypergeometric function was investigated concerning many aspects such as its univalence ([4], [5]) its applications on certain classes of univalent functions [8] and it was used in defining new operators [9].

Many basically equivalent definitions of fractional computation have been given in literature ((cf.)e.g.,[10] and ([12], p. 45)). We state the following definitions due to Owa and Srivastava [6] which have been used rather frequently in the theory of analytic functions (see also [3]).

Definition 1.1. The fractional integral of order ϑ is defined, for a function (z) , by

$$D_z^{-\vartheta} u(z) = \frac{1}{\hbar(\vartheta)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\vartheta}} d\zeta, \quad (\vartheta > 0) \tag{1.5}$$

and the fractional derivative of order ς is defined, for a function $u(z)$, by

$$D_z^{\vartheta} u(z) = \frac{1}{\hbar(1-\vartheta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\vartheta}} d\zeta, \quad (0 \leq \vartheta < 1), \tag{1.6}$$

where $u(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\vartheta-1}$ involved in (1.5) (and that of $(z-\zeta)^{-\vartheta}$ involved in (1.6) is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.2. Under the hypotheses of Definition 1.1, the fractional derivative of order $n+\vartheta$ is defined by

$$D_z^{n+\vartheta} u(z) = \frac{d^n}{dz^n} D_z^{\vartheta} u(z), \quad (0 \leq \vartheta < 1; n \in \mathrm{N}_0 = \mathrm{N} \cup \{0\}). \tag{1.7}$$

With the aid of the above definitions, Owa and Srivastava [6] defined the fractional operator $\mathcal{J}_z^{\vartheta}$ by

$$\mathcal{J}_z^{\vartheta} u(z) = \hbar(2-\vartheta) z^{\vartheta} D_z^{\vartheta} u(z), \quad (\vartheta \neq 2,3,4,\cdots)$$

$$\mathcal{J}_z^{\vartheta} u(z) = z + \sum_{n=2}^{\infty} \phi(\vartheta, n) a_n z^n \tag{1.8}$$

$$\text{where } \phi(\vartheta, n) = \frac{\hbar(n+1)\hbar(2-\vartheta)}{\hbar(n-\vartheta+1)} \tag{1.9}$$

$$\text{and } \phi(\vartheta, 2) = \frac{2}{(2-\vartheta)}.$$

Now, by making use of the linear operator $\mathcal{J}_z^{\vartheta} u$, we define a new subclass of functions belonging to the class A.

Definition 1.3. For $0 \leq \hbar < 1, 0 \leq \sigma < 1, 0 < \varsigma < 1,$ and $0 \leq \vartheta < 1$, we let $TS(\hbar, \sigma, \varsigma, \vartheta)$ be the subclass of u consisting of functions of the form (1.4) and its geometrical condition satisfy

$$\left| \frac{\hbar\left(\left(\mathcal{J}_z^{\vartheta} u(z)\right)' - \frac{\mathcal{J}_z^{\vartheta} u(z)}{z}\right)}{\sigma\left(\mathcal{J}_z^{\vartheta} u(z)\right)' + (1-\hbar)\frac{\mathcal{J}_z^{\vartheta} u(z)}{z}} \right| < \varsigma, \quad z \in \mathbb{U}$$

where $\mathcal{J}_z^{\vartheta}$, is given by (1.8).

2. Coefficient Inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $TS(\hbar, \sigma, \varsigma, \vartheta)$.

Theorem 2.1. Let the function u be defined by (1.4). Then $u \in TS(\hbar, \sigma, \varsigma, \vartheta)$ if and only if

$$\sum_{n=2}^{\infty} [\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)] \phi(n, \vartheta) a_n \leq \varsigma\left(\sigma + (1-\hbar)\right), \tag{2.1}$$

where $0 < \varsigma < 1, 0 \leq \hbar < 1, 0 \leq \sigma < 1,$ and $0 \leq \vartheta < 1$. The result (2.1) is sharp for the function

$$u(z) = z - \frac{\varsigma\left(\sigma + (1-\hbar)\right)}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi(n, \vartheta)} z^n, \quad n \geq 2.$$

Proof. Suppose that the inequality (2.1) holds true and $|z| = 1$. Then we obtain

$$\left|\hbar\left(\left(\mathcal{J}_z^{\vartheta}u(z)\right)' - \frac{\mathcal{J}_z^{\vartheta}u(z)}{z}\right)\right| - \varsigma\left|\sigma\left(\mathcal{J}_z^{\vartheta}u(z))' + (1-\hbar)\frac{\mathcal{J}_z^{\vartheta}u(z)}{z}\right)\right|$$

$$= \left|-\hbar\sum_{n=2}^{\infty}(n-1)\,\phi(n,\vartheta)a_n z^{n-1}\right|$$

$$-\varsigma\left|\sigma + (1-\hbar) - \sum_{n=2}^{\infty}(n\sigma + 1 - \hbar)\,\phi(n,\vartheta)a_n z^{n-1}\right|$$

$$\leq \sum_{n=2}^{\infty}[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\,\phi(n,\vartheta)a_n - \varsigma\big(\sigma + (1-\hbar)\big)$$

$$\leq 0.$$

Hence, by maximum modulus principle,$u \in TS(\hbar,\sigma,\varsigma,\vartheta)$. Now assume that $u \in TS(\hbar,\sigma,\varsigma,\vartheta)$ so that

$$\left|\frac{\hbar\left(\left(\mathcal{J}_z^{\vartheta}u(z)\right)' - \frac{\mathcal{J}_z^{\vartheta}u(z)}{z}\right)}{\sigma\big(\mathcal{J}_z^{\vartheta}u(z)\big)' + (1-\hbar)\frac{\mathcal{J}_z^{\vartheta}u(z)}{z}}\right| < \varsigma, \quad z \in \mathbb{U}$$

Hence

$$\left|\hbar\left(\left(\mathcal{J}_z^{\vartheta}u(z)\right)' - \frac{\mathcal{J}_z^{\vartheta}u(z)}{z}\right)\right| < \varsigma\left|\sigma\left(\mathcal{J}_z^{\vartheta}u(z))' + (1-\hbar)\frac{\mathcal{J}_z^{\vartheta}u(z)}{z}\right)\right|.$$

Therefore, we get

$$\left|-\sum_{n=2}^{\infty}\hbar\,(n-1)\phi(n,\vartheta)a_n z^{n-1}\right|$$

$$< \ \varsigma\left|\sigma + (1-\hbar) - \sum_{n=2}^{\infty}(n\sigma + 1 - \hbar)\,\phi(n,\vartheta)a_n z^{n-1}\right|.$$

Thus

$$\sum_{n=2}^{\infty}[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\,\phi(n,\vartheta)a_n \leq \varsigma\big(\sigma + (1-\hbar)\big)$$

and this completes the proof.

Corollary 2.1. Let the function $u \in TS(\hbar,\sigma,\varsigma,\vartheta)$.Then

$$a_n \leq \frac{\varsigma\big(\sigma + (1-\hbar)\big)}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi(n,\vartheta)}z^n, \quad n \geq 2.$$

3. Distortion and Covering Theorem

We introduce the growth and distortion theorems for the functions in the class $TS(\hbar,\sigma,\varsigma,\vartheta)$

Theorem 3.1. Let the function $u \in TS(\hbar,\sigma,\varsigma,\vartheta)$. Then

$$|z| - \frac{\varsigma\big(\sigma + (1-\hbar)\big)}{\phi(2,\vartheta)[\hbar + \varsigma(2\sigma + 1 - \hbar)]}|z|^2 \leq |u(z)|$$

$$\leq |z| + \frac{\varsigma\big(\sigma + (1-\hbar)\big)}{\phi(2,\vartheta)[\hbar + \varsigma(2\sigma + 1 - \hbar)]}|z|^2.$$

The result is sharp and attained

$$u(z) = z - \frac{\varsigma\big(\sigma + (1-\hbar)\big)}{\phi(2,\vartheta)[\hbar + \varsigma(2\sigma + 1 - \hbar)]}z^2.$$

Proof.

$$|u(z)| = \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \le |z| + \sum_{n=2}^{\infty} a_n |z|^n$$
$$\le |z| + |z|^2 \sum_{n=2}^{\infty} a_n .$$

By Theorem 2.1, we get

$$\sum_{n=2}^{\infty} a_n \le \frac{\varsigma\big(\sigma+(1-\hbar)\big)}{[\hbar+\varsigma(2\sigma+1-\hbar)]\phi(n,\vartheta)}. \qquad (3.1)$$

Thus

$$|u(z)| \le |z| + \frac{\varsigma\big(\sigma+(1-\hbar)\big)}{\phi(2,\vartheta)[\hbar+\varsigma(2\sigma+1-\hbar)]}|z|^2.$$

Also

$$|u(z)| \ge |z| - \sum_{n=2}^{\infty} a_n |z|^n$$
$$\ge |z| - |z|^2 \sum_{n=2}^{\infty} a_n$$
$$\ge |z| - \frac{\varsigma\big(\sigma+(1-\hbar)\big)}{\phi(2,\vartheta)[\hbar+\varsigma(2\sigma+1-\hbar)]}|z|^2.$$

Theorem 3.2. Let $u \in TS(\hbar,\sigma,\varsigma,\vartheta)$. Then

$$1 - \frac{2\varsigma\big(\sigma+(1-\hbar)\big)}{\phi(2,\vartheta)[\hbar+\varsigma(2\sigma+1-\hbar)]}|z| \le |u'(z)| \le 1 + \frac{2\varsigma\big(\sigma+(1-\hbar)\big)}{\phi(2,\vartheta)[\hbar+\varsigma(2\sigma+1-\hbar)]}|z|$$

with equality for

$$u(z) = z - \frac{2\varsigma\big(\sigma+(1-\hbar)\big)}{\phi(2,\vartheta)[\hbar+\varsigma(2\sigma+1-\hbar)]}z^2.$$

Proof. Notice that

$$\begin{aligned} &\phi(2,\vartheta)[\hbar+\varsigma(2\sigma+1-\hbar)]\textstyle\sum_{n=2}^{\infty} n\, a_n \\ &\le \textstyle\sum_{n=2}^{\infty} n\,[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi(n,\vartheta)a_n \\ &\le \varsigma\big(\sigma+(1-\hbar)\big), \end{aligned} \qquad (3.2)$$

from Theorem 2.1. Thus

$$\begin{aligned} |u'(z)| &= \left|1 - \textstyle\sum_{n=2}^{\infty} n\, a_n z^{n-1}\right| \\ &\le 1 + \textstyle\sum_{n=2}^{\infty} n\, a_n |z|^{n-1} \\ &\le 1 + |z| \textstyle\sum_{n=2}^{\infty} n\, a_n \\ &\le 1 + |z| \frac{2\varsigma(\sigma+(1-\hbar))}{\phi(2,\vartheta)[\hbar+\varsigma(2\sigma+1-\hbar)]}. \end{aligned} \qquad (3.3)$$

On the other hand

$$\begin{aligned} |u'(z)| &= \left|1 - \textstyle\sum_{n=2}^{\infty} n\, a_n z^{n-1}\right| \\ &\ge 1 - \textstyle\sum_{n=2}^{\infty} n\, a_n |z|^{n-1} \\ &\ge 1 - |z| \textstyle\sum_{n=2}^{\infty} n\, a_n \\ &\ge 1 - |z| \frac{2\varsigma(\sigma+(1-\hbar))}{\phi(2,\vartheta)[\hbar+\varsigma(2\sigma+1-\hbar)]}. \end{aligned} \qquad (3.4)$$

Combining (3.3) and (3.4), we get the result.

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $TS(\hbar,\sigma,\varsigma,\vartheta)$.

Theorem 4.1. Let $u \in TS(\hbar, \sigma, \varsigma, \vartheta)$. Then u is starlike in $|z| < R_1$ of order δ, $0 \le \delta < 1$, where

$$R_1 = \inf_n \left\{\frac{(1-\delta)\big(\hbar(n-1)+\varsigma(n\sigma+1-\hbar)\big)\phi(n,\vartheta)}{(n-\delta)\varsigma\big(\sigma+(1-\hbar)\big)}\right\}^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.1}$$

Proof. u is starlike of order $\delta, 0 \le \delta < 1$ if

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \delta.$$

Thus it is enough to show that

$$\left|\frac{zu'(z)}{u(z)} - 1\right| = \left|\frac{-\sum_{n=2}^{\infty}(n-1)\, a_n z^{n-1}}{1-\sum_{n=2}^{\infty} a_n\, z^{n-1}}\right| \le \frac{\sum_{n=2}^{\infty}(n-1)\, a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n\, |z|^{n-1}}.$$

Thus

$$\left|\frac{zu'(z)}{u(z)} - 1\right| \le 1-\delta \text{ if } \sum_{n=2}^{\infty}\frac{(n-\delta)}{(1-\delta)} a_n |z|^{n-1} \le 1. \tag{4.2}$$

Hence by Theorem 2.1, (4.2) will be true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} \le \frac{\big(\hbar(n-1)+\varsigma(n\sigma+1-\hbar)\big)\phi(n,\vartheta)}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z| \le \left[\frac{(1-\delta)\big(\hbar(n-1)+\varsigma(n\sigma+1-\hbar)\big)\phi(n,\vartheta)}{(n-\delta)\varsigma\big(\sigma+(1-\hbar)\big)}\right]^{\frac{1}{n-1}}, n \ge 2. \tag{4.3}$$

The Theorem 4.1 follows easily from (4.3).

Theorem 4.2. Let $u \in TS(\hbar, \sigma, \varsigma, \vartheta)$. Then u is convex in $|z| < R_2$ of order $\delta, 0 \le \delta < 1$, where

$$R_2 = \inf_n \left\{\frac{(1-\delta)\big(\hbar(n-1)+\varsigma(n\sigma+1-\hbar)\big)\phi(n,\vartheta)}{n(n-\delta)\varsigma\big(\sigma+(1-\hbar)\big)}\right\}^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.4}$$

Proof. u is convex of order $\delta, 0 \le \delta < 1$ if

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \delta.$$

Thus it is enough to show that

$$\left|\frac{zu''(z)}{u'(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} n\,(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} n\, a_n z^{n-1}}\right| \le \frac{\sum_{n=2}^{\infty} n\,(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} n\, a_n |z|^{n-1}}.$$

Thus

$$\left|\frac{zu''(z)}{u'(z)}\right| \le 1-\delta \text{ if } \sum_{n=2}^{\infty}\frac{n(n-\delta)}{(1-\delta)} a_n |z|^{n-1} \le 1. \tag{4.5}$$

Hence by Theorem 2.1, (4.5) will be true if

$$\frac{n(n-\delta)}{1-\delta}|z|^{n-1} \le \frac{\big(\hbar(n-1)+\varsigma(n\sigma+1-\hbar)\big)\phi(n,\vartheta)}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z| \le \left[\frac{(1-\delta)\big(\hbar(n-1)+\varsigma(n\sigma+1-\hbar)\big)\phi(n,\vartheta)}{n(n-\delta)\varsigma\big(\sigma+(1-\hbar)\big)}\right]^{\frac{1}{n-1}}, n \ge 2. \tag{4.6}$$

The theorem follows easily from (4.6).

Theorem 4.3. Let $u \in TS(\hbar, \sigma, \varsigma, \vartheta)$. Then u is close-to-convex in $|z| < R_3$ of order δ, $0 \le \delta < 1$, where

$$R_3 = \inf_n \left\{\frac{(1-\delta)\big(\hbar(n-1)+\varsigma(n\sigma+1-\hbar)\big)\phi(n,\vartheta)}{n\varsigma\big(\sigma+(1-\hbar)\big)}\right\}^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.7}$$

Proof. u is close-to-convex of order $\delta, 0 \le \delta < 1$ if

$$\Re\{u'(z)\} > \delta.$$

Thus it is enough to show that

$$|u'(z)-1| = \left|-\sum_{n=2}^{\infty} n\, a_n z^{n-1}\right| \le \sum_{n=2}^{\infty} n\, a_n |z|^{n-1}.$$

Thus

$$|u'(z)-1| \le 1-\delta \text{ if } \sum_{n=2}^{\infty} \frac{n}{(1-\delta)} a_n |z|^{n-1} \le 1. \tag{4.8}$$

Hence by Theorem 2.1, (4.8) will be true if

$$\frac{n}{1-\delta}|z|^{n-1} \le \frac{\big(\hbar(n-1)+\varsigma(n\sigma+1-\hbar)\big)\phi(n,\vartheta)}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z| \le \left[\frac{(1-\delta)\big(\hbar(n-1)+\varsigma(n\sigma+1-\hbar)\big)\phi(n,\vartheta)}{n\varsigma\big(\sigma+(1-\hbar)\big)}\right]^{\frac{1}{n-1}}, n \ge 2. \tag{4.9}$$

The theorem follows easily from (4.9).

5. Extreme Points

In the following theorem, we obtain extreme points for the class $TS(\hbar,\sigma,\varsigma,\vartheta)$.

Theorem 5.1. Let $u_1(z) = z$ and

$$u_n(z) = z - \frac{\varsigma\big(\sigma+(1-\hbar)\big)}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi(n,\vartheta)} z^n, \text{ for } n = 2,3,\cdots.$$

Then $u \in TS(\hbar,\sigma,\varsigma,\vartheta)$ if and only if it can be expressed in the form

$$u(z) = \sum_{n=1}^{\infty} \theta_n\, u_n(z), \text{ where } \theta_n \ge 0 \text{ and } \sum_{n=1}^{\infty} \theta_n = 1.$$

Proof. Assume that $u(z) = \sum_{n=1}^{\infty} \theta_n\, u_n(z)$, hence we get

$$u(z) = z - \sum_{n=2}^{\infty} \frac{\varsigma\big(\sigma+(1-\hbar)\big)\theta_n}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi(n,\vartheta)} z^n.$$

Now, $u \in TS(\hbar,\sigma,\varsigma,\vartheta)$, since

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi(n,\vartheta)}{\varsigma\big(\sigma+(1-\hbar)\big)} \times \frac{\varsigma\big(\sigma+(1-\hbar)\big)\theta_n}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi(n,\vartheta)} = \sum_{n=2}^{\infty} \theta_n = 1-\theta_1 \le 1.$$

Conversely, suppose $u \in TS(\hbar,\sigma,\varsigma,\vartheta)$. Then we show that u can be written in the form $\sum_{n=1}^{\infty} \theta_n\, u_n(z)$.
Now $u \in TS(\hbar,\sigma,\varsigma,\vartheta)$ implies from Theorem 2.1

$$a_n \le \frac{\varsigma\big(\sigma+(1-\hbar)\big)}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi(n,\vartheta)}.$$

Setting $\theta_n = \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi(n,\vartheta)}{\varsigma\big(\sigma+(1-\hbar)\big)} a_n, n = 2,3,\cdots$
and $\theta_1 = 1 - \sum_{n=2}^{\infty} \theta_n$, we obtain $u(z) = \sum_{n=1}^{\infty} \theta_n\, u_n(z)$.

6. Hadamard product

In the following theorem, we obtain the convolution result for functions belongs to the class $TS(\hbar,\sigma,\varsigma,\vartheta)$.

Theorem 6.1. Let $u, g \in TS(\hbar,\sigma,\varsigma,\vartheta)$. Then $u * g \in TS(\hbar,\sigma,\zeta,\vartheta)$ for

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } (u * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n,$$

where

$$\zeta \geq \frac{\varsigma^2\big(\sigma + (1-\hbar)\big)\hbar(n-1)}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]^2\phi(n,\vartheta) - \varsigma^2\big(\sigma + (1-\hbar)\big)(n\sigma + 1 - \hbar)}.$$

Proof. $u \in TS(\hbar, \sigma, \varsigma, \vartheta)$ and so

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi(n,\vartheta)}{\varsigma\big(\sigma+(1-\hbar)\big)} a_n \leq 1, \tag{6.1}$$

and

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi(n,\vartheta)}{\varsigma\big(\sigma+(1-\hbar)\big)} b_n \leq 1. \tag{6.2}$$

We have to find the smallest number ζ such that

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\phi(n,\vartheta)}{\zeta\big(\sigma+(1-\hbar)\big)} a_n b_n \leq 1. \tag{6.3}$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\phi(n,\vartheta)}{\varsigma\big(\sigma+(1-\hbar)\big)} \sqrt{a_n b_n} \leq 1. \tag{6.4}$$

Therefore it is enough to show that

$$\frac{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\phi(n,\vartheta)}{\zeta\big(\sigma + (1-\hbar)\big)} a_n b_n \leq \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi(n,\vartheta)}{\varsigma\big(\sigma + (1-\hbar)\big)} \sqrt{a_n b_n}.$$

That is

$$\sqrt{a_n b_n} \leq \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\zeta}{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\varsigma}. \tag{6.5}$$

From (6.4)

$$\sqrt{a_n b_n} \leq \frac{\varsigma\big(\sigma + (1-\hbar)\big)}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi(n,\vartheta)}.$$

Thus it is enough to show that

$$\frac{\varsigma\big(\sigma + (1-\hbar)\big)}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi(n,\vartheta)} \leq \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\zeta}{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\varsigma},$$

which simplifies to

$$\zeta \geq \frac{\varsigma^2\big(\sigma + (1-\hbar)\big)\hbar(n-1)}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]^2\phi(n,\vartheta) - \varsigma^2\big(\sigma + (1-\hbar)\big)(n\sigma + 1 - \hbar)}.$$

7. Closure Theorems

We shall prove the following closure theorems for the class $TS(\hbar, \sigma, \varsigma, \vartheta)$.

Theorem 7.1. Let $u_j \in TS(\hbar, \sigma, \varsigma, \vartheta), j = 1,2, \ldots, s$. Then

$$g(z) = \sum_{j=1}^{s} c_j u_j(z) \in TS(\hbar, \sigma, \varsigma, \vartheta)$$

For $u_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n$, where $\sum_{j=1}^{s} c_j = 1$.

Proof.

$$g(z) = \sum_{j=1}^{s} c_j\, u_j(z)$$
$$= z - \sum_{n=2}^{\infty}\sum_{j=1}^{s} c_j\, a_{n,j} z^n$$
$$= z - \sum_{n=2}^{\infty} e_n\, z^n,$$

where $e_n = \sum_{j=1}^{s} c_j\, a_{n,j}$. Thus $g(z) \in TS(\hbar, \sigma, \varsigma, \vartheta)$ if

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi(n,\vartheta)}{\varsigma\big(\sigma + (1-\hbar)\big)} e_n \le 1,$$

that is, if

$$\sum_{n=2}^{\infty}\sum_{j=1}^{s} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi(n,\vartheta)}{\varsigma\big(\sigma + (1-\hbar)\big)} c_j a_{n,j}$$
$$= \sum_{j=1}^{s} c_j \sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\phi(n,\vartheta)}{\varsigma\big(\sigma + (1-\hbar)\big)} a_{n,j}$$
$$\le \sum_{j=1}^{s} c_j = 1.$$

Theorem 7.2. Let $u, g \in TS(\hbar, \sigma, \varsigma, \vartheta)$. Then

$$h(z) = z - \sum_{n=2}^{\infty} (a_n^2 + b_n^2)\, z^n \in TS(\hbar, \sigma, \varsigma, \vartheta), \text{ where}$$

$$\zeta \ge \frac{2\hbar(n-1)\varsigma^2\big(\sigma + (1-\hbar)\big)}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]^2\phi(n,\vartheta) - 2\varsigma^2\big(\sigma + (1-\hbar)\big)(n\sigma + 1 - \hbar)}.$$

Proof. Since $u, g \in TS(\hbar, \sigma, \varsigma, \vartheta)$, so Theorem 2.1 yields

$$\sum_{n=2}^{\infty} \left[\frac{\big(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)\big)\phi(n,\vartheta)}{\varsigma\big(\sigma + (1-\hbar)\big)} a_n\right]^2 \le 1$$

and

$$\sum_{n=2}^{\infty} \left[\frac{\big(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)\big)\phi(n,\vartheta)}{\varsigma\big(\sigma + (1-\hbar)\big)} b_n\right]^2 \le 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty} \frac{1}{2}\left[\frac{\big(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)\big)\phi(n,\vartheta)}{\varsigma\big(\sigma + (1-\hbar)\big)}\right]^2 (a_n^2 + b_n^2) \le 1. \tag{7.1}$$

But $h(z) \in TS(\hbar, \sigma, \zeta, q, m)$, if and only if

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\phi(n,\vartheta)}{\zeta\big(\sigma + (1-\hbar)\big)} (a_n^2 + b_n^2) \le 1, \tag{7.2}$$

where $0 < \zeta < 1$, however (7.1) implies (7.2) if

$$\frac{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\phi(n,\vartheta)}{\zeta\big(\sigma + (1-\hbar)\big)}$$
$$\le \frac{1}{2}\left[\frac{\big(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)\big)\phi(n,\vartheta)}{\varsigma\big(\sigma + (1-\hbar)\big)}\right]^2.$$

Simplifying, we get

$$\zeta \geq \frac{2\hbar(n-1)\varsigma^2\big(\sigma+(1-\hbar)\big)}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]^2\phi(n,\vartheta)-2\varsigma^2\big(\sigma+(1-\hbar)\big)(n\sigma+1-\hbar)}.$$

8. Conclusion

This research has introduced a new fractional calclus operator and studied some basic properties of geometric function theory. Accordingly, some results related to coefficient estimates, growth and distortion properties, closure theorems and hadamard product have also been considered, inviting future research for this field of study.

REFERENCES:

[1] N.E.,Cho, S.Y. Woo and S. Owa, Uniform convexity properties for hypergeometric functions, *Fract. Calc. Appl. Anal.*, 5 (3) (2002), 303 - 313.

[2] L.De Branges, A proof of the Bieberbach conjecture, *Acta Math.*, 154 (1-2) (1985), 137 - 152.

[3] Y. C. Kim and H.M. Srivastava, Fractional integral and other linear operators asso ciated with the Gaussian hypergeometric function, *Complex Variables Theory Appl.* , 34 (1997), 293 - 312.

[4] E.P.Merkes and W.T. Scott, Starlike hypergeometric functions, *Proc. Amer. Math. Soc.*, 12 (1961), 885 - 888.

[5] A.O. Mostafa, A study on starlike and convex properties for hypergeometric functions, *J. Inequal. Pure Appl. Math.*, 10 (3) (2009), Article 87, 8 pages.

[6] S. Owa and H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Can. J. Math.* , 39 (1987), 1057-1077.

[7] C.Pommerenke, *Univalent Functions,* Vandenhoeck and Ruprecht, Gottingen, 1975.

[8] H.Silverman, Starlike and convexity properties for hypergeometric functions, *J. Math. Anal. Appl.* 172 (2)(1993), 574 - 581.

[9] H.M. Srivastava, G. Murugusundaramoorthy and S. Sivasubramanian, Hypergeometric functions in the parabolic starlike and uniformly convex domains, *Integral Transforms Spec. Funct.* , 18 (7-8) (2007), 511 - 520.

[10] S.G.Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integral and Derivatives, Theory and Applications,* Gordon and Breach, New York, Philadelphia, London, Paris, Montreux, Toronto and Melbourne, 1993.

[11] H.Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, 51 (1975), 109-116.

[12] H.M Srivastava and R.G. Buschman, *Theory and Applications of Convolution Integral Equations,* Kluwer Academic Publishers, dordrecht, Boston and London, 1992.