Electrostatic Potential in Spherical Region

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Abstract:
In this paper we have utilized multivariable I-functions and Legendre series to obtain harmonic function satisfying the partial differential equation and boundary conditions of the problems on electrostatic potential in a region of space that is free from the electric charges. Special cases include the result given by Divera.

Key Words: Multivariable $\bar{I}$ –functions, Legendre's series, Harmonic function.

1. Introduction:

This paper deals with the electrostatic potential problem in a spherical region that is free from electric charges. The problem is solved using Legendre polynomials and multivariable I-functions. The multivariable I-function was introduced by Prathima and Vasudevan Nambisan. In this paper, we employ a special case of the multivariable I-function, known as the $\bar{I}$ -function.

Notations and Results used:

$(a)_n$ stands for $a(a+1)..(a+n-1)$

${}_1(a_j;\alpha_j,A_j)_p$ stands for $(a_1;\alpha_1,A_1),(a_2;\alpha_2,A_2),...,(a_p;\alpha_p,A_p)$

$\bar{I}$ -function of r-variables defined by Prathima and Vasudevan Nambisan [7] is given by,

$$\bar{I}[z_1,.....,z_r] = I^{0,N:\ m_1,n_1;...;m_r,n_r}_{P,Q:\ p_1,q_1;...;p_r,q_r}\left[\begin{array}{c|c} z_1 & I_1 \\ \vdots & \\ z_r & I_2 \end{array}\right]$$

$$= \frac{1}{(2\pi\ \omega)^r}\int_{L_1}...\int_{Lr}\bar{\theta}_1(s_1)...\ \bar{\theta}_r(s_r)\ \varphi\ (s_1,...,s_r)\ z_1^{s_1}...z_r^{s_r}\,ds_1...ds_r, \qquad (1.1)$$ where,

$$I_1 = {}_1(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)};A_j)_P : {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};...;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}$$

$$I_2 = {}_1(b_j;\beta_j^{(1)},...,\beta_j^{(r)};B_j)_Q : {}_1\left(d_j{}^{(1)},\delta_j{}^{(1)};1\right)_{m_1};\ {}_{m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};...;\ {}_1\left(d_j{}^{r},\delta_j{}^{r};1\right)_{m_r};\ {}_{m_r+1}(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}$$

(1.2)

$$\overline{\theta}_i(s_i) = \frac{\prod_{j=1}^{m_i}\Gamma\ (d_j^{(i)} - \delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)},$$

(1.3)

where , $i = 1,2,..r.$

The integral (1.1) converges absolutely if $\left|\ \arg(z_i)\ \right| < \frac{1}{2}\Delta'_i\,\pi, i = 1,2,...,r,$

(1.4)

where,

$$\Delta'_i = \left(-\sum_{j=n+1}^{P}A_j\alpha_j^{(i)} - \sum_{j=1}^{Q}B_j\beta_j^{(i)} + \sum_{j=1}^{m_i}\ \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i}D_j^{(i)}\delta_j^{(i)}\ + \sum_{j=1}^{n_i}C_j^{(i)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i}C_j^{(i)}\gamma_j^{(i)}\right) > 0,\ i = 1,2,\ldots,r.$$

(1.5)

The results given by Mathai and Saxena [3] is,

$$\int_{-1}^{1}(1-u)^{\sigma}\,P_k(u)\,du\ = 2^{\sigma+1}\frac{\Gamma(1+\sigma)\Gamma(k-\sigma)}{\Gamma(1+\sigma+k)\Gamma(-\sigma)}$$

(1.6)

2.Preliminary Result

$$\int_{-1}^{1}(1-u)^{\sigma}\,P_k(x)\overline{I}^{0,N:\ m_1,n_1;\ldots;m_r,n_r}_{P,Q:\ p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c|c} z_1\left(\frac{1-u}{2}\right)^{\lambda} & I_1 \\ z_2 & \\ \vdots & \\ z_r & I_2 \end{array}\right] = 2^{\sigma+1}\overline{I}^{0,N:\ m_1+1,n_1+1;\ldots;m_r,n_r}_{P,Q:\ p_1+2,q_1+2;\ldots;p_r,q_r}\left[\begin{array}{c|c} \frac{z_1}{2^{\lambda}} & I_3 \\ z_2 & \\ \vdots & \\ z_r & I_4 \end{array}\right]$$

(2.1)

where $P_k(x)$ is the Legendre polynomial and

$$I_1 = {}_1\left(a_j,\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_P : {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\ldots;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}$$

(2.2)

$$I_2 = {}_1\left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_Q;{}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1};{}_{m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\ldots;{}_1\left(d_j{}^r,\delta_j{}^r;1\right)_{m_r};{}_{m_r+1}(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}$$

(2.3)

$$I_3 = {}_1\left(a_j,\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_P : (\sigma,\lambda;1);{}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};(-\sigma,-\lambda;1);\ldots;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}$$

(2.4)

$$I_4 = {}_1\left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_Q;(k-\sigma,-\lambda;1)\,{}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1};{}_{m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};(-\sigma-k-1,\lambda);\ldots;$$
$${}_1\left(d_j{}^r,\delta_j{}^r;1\right)_{m_r};{}_{m_r+1}(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}$$

(2.5)

provided $\lambda > 0$ and $\left|\ \arg(z_i)\ \right| < \frac{1}{2}\Delta'_i\,\pi, i = 1,2,...,r$ and Δ'_i *given by* (1.5)

Proof:

Consider the integral $$\int_{-1}^{1}(1-u)^{\sigma} P_k(u)\bar{I}_{P,Q:\ p_1,q_1;\ldots;p_r,q_r}^{0,N:\ m_1,n_1;\ldots;m_r,n_r}\left[\begin{array}{c|c} z_1\left(\frac{1-u}{2}\right)^{\lambda} & I_1 \\ z_2 & \\ \vdots & \\ z_r & I_2 \end{array}\right] \quad (2.6)$$

Expressing (2.6) as a contour integral using (1.1), the expression in (2.6) reduces to:

$$\int_{-1}^{1}(1-u)^{\sigma} P_k(u)\frac{1}{(2\pi i)^r}\Big(\int_{L_1}\ldots\int_{Lr}(\bar{\theta}_1(s_1)\ldots\bar{\theta}_r(s_r)\ \varphi(s_1,\ldots,s_r)\left(z_1\left(\frac{1-u}{2}\right)^{\lambda}\right)^{s_1}(z_2)^{s_2}\ldots(z_r)^{s_r}\,ds_1\ldots ds_r\Big)\,du \quad (2.7)$$

Now, changing the order of integration, which is justified under the given conditions (2.7) reduces to:

$$\frac{1}{(2\pi i)^r}\int_{L_1}\ldots\int_{Lr}\bar{\theta}_1(s_1)\ldots\bar{\theta}_r(s_r)\ \varphi(s_1,\ldots,s_r)\left(\frac{z_1}{2^{\lambda}}\right)^{s_1} z_2\ldots z_r^{s_r}\left(\int_{-1}^{1}(1-u)^{\sigma+\lambda s_1}P_k(u)\,du\right)ds_1\ldots ds_r \quad (2.8)$$

Now evaluating inner integrals with the help of (1.6), the expression (2.8) becomes,

$$\frac{1}{(2\pi i)^r}\int_{L_1}\ldots\int_{Lr}\Big(\bar{\theta}_1(s_1)\ \ldots\bar{\theta}_r(s_r)\ \varphi(s_1,\ldots,s_r)\left(\frac{z}{2^{\lambda}}\right)^{s_1} z_2^{s_2}\ldots z_r^{s_r} \times\frac{\Gamma(1+\sigma+\lambda s_1)\Gamma(k-\sigma-\lambda s_1)}{\Gamma(1+\sigma+k+\lambda s_1)\Gamma(-\sigma-\lambda s_1)}\Big)\,ds_1\ldots ds_r \quad (2.9)$$

Interpretting (2.9) with the help of (1.1), the expression (2.9) becomes:

$$= 2^{\sigma+1}\bar{I}_{P,Q:\ p_1+2,q_1+2;p_2,q_2;\ldots;p_r,q_r}^{0,N:\ m_1+1,n_1+1;m_2,n_2\ldots;m_r,n_r}\left[\begin{array}{c|c} \frac{z_1}{2^{\lambda}} & I_3 \\ z_2 & \\ \vdots & \\ z_r & I_4 \end{array}\right],$$

I_3 *and* I_4 *are given by* (2.4) *and* (2.5) provided $\lambda > 0$

and $\left|\ \arg(z_i)\ \right| < \frac{1}{2}\Delta'_i\,\pi, i = 1,2,\ldots,r$

where,

Δ'_i *given by* (1.5)

3. Electrostatic Potential in Spherical Regions

Here we shall utilize I- function and Legendre's Series to obtain the harmonic function v representing potential in the domain r < c such that v assumes prescribed values $F(\theta)$ On the spherical surface $r = c$.

If we take $F(\theta) = f(x)$ then $v(r,x)$ satisfies the equation

$$r\frac{\partial^2(rv)}{\partial r^2} + \frac{d}{dx}\left((1-x^2)\frac{dv}{dx}\right) = 0, r < c, -1 < x < 1$$

and boundary condition of the problem

$$\lim_{r \to c} v(r,x) = f(x), \quad r < c, -1 < x < c$$

where v is continuous everywhere interior to the sphere and bounded and

w is the harmonic function in the unbounded domain $r > c$, exterior to the spherical surface $r = c$ and rw is bounded for large values of x and for all $x(-1 < x < 1)$

Now we derive the solution of our boundary value problem for

$$f(x) = (1-x)^{\sigma}\, \bar{I}^{0,N:\ m_1,n_1;\ldots;m_r,n_r}_{P,Q:\ p_1,q_1;\ldots;p_r,q_r} \left[\begin{array}{c|c} z_1\left(\frac{1-x}{2}\right)^{\lambda} & I_1 \\ z_2 & \\ \vdots & \\ z_r & I_2 \end{array}\right] \tag{3.1}$$

Provided $\lambda > 0$ and I_1 and I_2 are given by (2.2) and (2.3) respectively.

Case 1: Solution of v(r,x) for r < c , interior to the sphere.
The solution is

$$v(r,x) = 2^{\sigma}\sum_{k=0}^{\infty}(2k+1)\left(\frac{r}{c}\right)^k P_k(x)\, \bar{I}^{0,N:\ m_1+1,n_1+1;.m_2,n_2..;m_r,n_r}_{P,Q:\ p_1+2,q_1+2;p_2,q_2;\ldots;p_r,q_r} \left[\begin{array}{c|c} \frac{z_1}{2^{\lambda}} & I_3 \\ z_2 & \\ \vdots & \\ z_r & I_4 \end{array}\right]$$

where $\lambda > 0, -1 < x < 1, r < c$

and I_3 and I_4 are given by (24) and (2.5) respectively.

and $\left|\ \arg(z_i)\ \right| < \frac{1}{2}\Delta'_i\,\pi, i = 1,2,\ldots,r$

where,

Δ'_i *given by* (1.6)

Proof:

On using $v(r,x)=\sum_{k=0}^{\infty} M_k \left(\frac{r}{c}\right)^k P_k(x), r \le c$

be the solution of our boundary value problem.

We find that the coefficients $v(c,x)=f(x)$

That is

$$f(x)=(1-x)^{\sigma}\ \overline{I}^{0,N:\ m_1,n_1;\ldots;m_r,n_r}_{P,Q:\ p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c|c} z_1\left(\frac{1-x}{2}\right)^{\lambda} & I_1 \\ z_2 & \\ \vdots & \\ z_r & I_2 \end{array}\right] \qquad (3.3)$$

$$=\sum_{k=0}^{\infty} M_k P_k(x) \qquad (3.4)$$

where $f(x)$ is continuous in the closed interval [-1,1] and has a piecewise continuous derivative there with Legendre series in (2.4) associated with $f(x)$ converges uniformly to $f(x)$ in $-1+\varepsilon \le x \le 1-\varepsilon, 0<\varepsilon<1$

Now multiply both sides of (3.4) by $P_m(x)$ and integrate between -1 and 1,we have

$$\int_{-1}^{1}(1-x)^{\sigma} P_m(x)\overline{I}^{0,N:\ m_1,n_1;\ldots;m_r,n_r}_{P,Q:\ p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{c|c} z_1\left(\frac{1-x}{2}\right)^{\lambda} & I_1 \\ z_2 & \\ \vdots & \\ z_r & I_2 \end{array}\right] = \sum_{k=0}^{\infty} M_k \int_{-1}^{1} P_k(x) P_m(x)\,dx \qquad (2.5)$$

Evaluate the left hand integral with the help of (2.1) and right hand integral using the orthogonality property of Legendre's polynomial, to get,

$$M_k = 2^{\sigma}\sum_{k=0}^{\infty}(2k+1)\overline{I}_{P,Q:\ p_1+2,q_1+2;p_2,q_2;\ldots;p_r,q_r}^{0,N:\ m_1+1,n_1+1;.m_2,n_2..;m_r,n_r}\left[\begin{array}{c|c} \frac{z_1}{2^{\lambda}} & I_3 \\ z_2 & \\ \vdots & \\ z_r & I_4 \\ & \end{array}\right] \quad (2.6)$$

where,

$$I_3 = {}_1\left(a_j,\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_P:(\sigma,\lambda;1);{}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};(-\sigma,-\lambda;1);\ldots;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}$$

$$I_4 = {}_1\left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_Q;(k-\sigma,-\lambda;1)\,{}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1};\,{}_{m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};$$

$$(-\sigma-k-1,\lambda);\ldots;\,{}_1\left(d_j{}^r,\delta_j{}^r;1\right)_{m_r};\,{}_{m_r+1}(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}$$

Putting (2.6) in (2.2), we get

$$v(r,x) = 2^{\sigma}\sum_{k=0}^{\infty}(2k+1)\left(\frac{r}{c}\right)^k P_k(x)\,\overline{I}_{P,Q:\ p_1+2,q_1+2;p_2,q_2;\ldots;p_r,q_r}^{0,N:\ m_1+1,n_1+1;.m_2,n_2..;m_r,n_r}\left[\begin{array}{c|c} \frac{z_1}{2^{\lambda}} & I_3 \\ z_2 & \\ \vdots & \\ z_r & I_4 \\ & \end{array}\right]$$

where $\lambda > 0, -1 < x < 1, r < c$

$$I_3 = {}_1\left(a_j,\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_P:(\sigma,\lambda;1);{}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};(-\sigma,-\lambda;1);\ldots;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}$$

$$I_4 = {}_1\left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_Q;(k-\sigma,-\lambda;1)\,{}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1};\,{}_{m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};$$

$$(-\sigma-k-1,\lambda);\ldots;\,{}_1\left(d_j{}^r,\delta_j{}^r;1\right)_{m_r};\,{}_{m_r+1}(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}$$

and $\left|\ \arg(z_i)\ \right| < \frac{1}{2}\Delta'_i\,\pi, i = 1,2,\ldots,r$

where,

$$\Delta'_i = \left(-\sum_{j=n+1}^{P}A_j\alpha_j^{(i)} - \sum_{j=1}^{Q}B_j\beta_j^{(i)} + \sum_{j=1}^{m_i}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i}D_j^{(i)}\delta_j^{(i)} + \sum_{j=1}^{n_i}C_j^{(i)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i}C_j^{(i)}\gamma_j^{(i)}\right) > 0 \quad \text{i} = 1,2,\ldots\text{r.}$$

Case 2: Solution of *w(r, x)* for *r > c* , exterior to the spherical surface

The solution is

$$w(r,x)=2^{\sigma}\sum_{k=0}^{\infty}(2k+1)\left(\frac{r}{c}\right)^{k+1}P_k(x)\,\overline{I}^{0,N:\ m_1+1,n_1+1;.m_2,n_2..;m_r,n_r}_{P,Q:\ p_1+2,q_1+2;p_2,q_2;...;p_r,q_r}\left[\begin{array}{c|c}\frac{z_1}{2^{\lambda}} & I_3\\ z_2 & \\ \vdots & \\ z_r & I_4\\ & \end{array}\right]$$

where $\lambda>0,-1<x<1,r\geq c$ (2.7)

Proof:

On using $$w(r,x)=\sum_{k=0}^{\infty}M_k\left(\frac{c}{r}\right)^{k+1}P_k(x),r\geq c \qquad (2.8)$$

be the solution of our boundary value problem.

The coefficients are such that $v(c,x)=f(x)$

That is,

$$f(x)=(1-x)^{\sigma}\,\overline{I}^{0,N:\ m_1,n_1;...;m_r,n_r}_{P,Q:\ p_1,q_1;...;p_r,q_r}\left[\begin{array}{c|c}z_1\left(\frac{1-x}{2}\right)^{\lambda} & I_1\\ z_2 & \\ \vdots & \\ z_r & I_2\\ & \end{array}\right] \qquad (2.9)$$

Now multiplying both sides of (2.9) by $P_m(x)$ and integrate between -1 and 1,we have

$$\int_{-1}^{1}(1-x)^{\sigma}P_m(x)\overline{I}^{0,N:\ m_1,n_1;...;m_r,n_r}_{P,Q:\ p_1,q_1;...;p_r,q_r}\left[\begin{array}{c|c}z_1\left(\frac{1-x}{2}\right)^{\lambda} & I_1\\ z_2 & \\ \vdots & \\ z_r & I_2\\ & \end{array}\right]=\sum_{k=0}^{\infty}M_k\int_{-1}^{1}P_k(x)P_m(x)\,dx \qquad (2.10)$$

Evaluate the left hand integral with the help of (2.1) and right hand integral using the orthogonality property of Legendre's polynomial, we get,

$$M_k = 2^{\sigma}\sum_{k=0}^{\infty}(2k+1)\bar{I}_{P,Q:\ p_1+2,q_1+2;p_2,q_2;\ldots;p_r,q_r}^{0,N:\ m_1+1,n_1+1;.m_2,n_2..;m_r,n_r}\begin{bmatrix} \frac{z_1}{2^{\lambda}} & I_3 \\ z_2 & \\ \vdots & \\ z_r & I_4 \\ & \end{bmatrix} \quad (2.11)$$

where,

$$I_3 = {}_1\left(a_j,\alpha_j^{(1)},\ldots\alpha_j^{(r)};A_j\right)_P:(\sigma,\lambda;1);{}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};(-\sigma,-\lambda;1);\ldots;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}$$

$$I_4 = {}_1\left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j\right)_Q;(k-\sigma,-\lambda;1)\,{}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1};{}_{m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};$$

$$(-\sigma-k-1,\lambda);\ldots;{}_1\left(d_j{}^r,\delta_j{}^r;1\right)_{m_r};{}_{m_r+1}(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}$$

Putting (2.11) in (2.8), to get the desired result.

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